

OFFICIAL COURSE STUDY GUIDE

Calculus 1: Full Course

Professor Leonard's complete, beloved calculus course.

SCIENCE & MATH · INTERMEDIATE · 6 MODULES · 41 HR 52 MIN

Course by Professor Leonard · Curated and published by MegaCourse

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Calculus 1: Full Course: The Complete Study Guide

2026 Edition

Published by MegaCourse · megacourse.app

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INTRODUCTION

About This Course

Welcome to Calculus 1: Full Course. This study guide supports your learning through Professor Leonard's complete and highly regarded calculus lectures. Calculus is a fundamental branch of mathematics, essential for understanding concepts across science, engineering, economics, and computer science.

Professor Leonard is known for his clear explanations and comprehensive approach. His video course provides a thorough grounding in the principles of single-variable calculus.

MegaCourse has organized this material into a structured learning path. We have added curated reading and resource lessons, a final knowledge-check quiz, and a clear framework to enhance your learning experience. This guide will help you navigate the course's six modules and 33 lessons.

By the end of this course, you will possess a strong foundation in limits, derivatives, and integrals. This knowledge prepares you for advanced mathematical studies and for solving complex real-world problems.

Course Facts

FORMAT	6 modules · 33 lessons
VIDEO INSTRUCTION	About 41 hr 52 min
LEVEL	Intermediate
CATEGORY	Science & Math
INSTRUCTOR	Professor Leonard
CERTIFICATE	Included with course completion on MegaCourse
COURSE HOME	megacourse.app/courses/calculus-1-full-course

What You Will Learn

- Understand the fundamental concepts of limits and continuity.
- Master the rules and applications of differentiation.
- Apply various techniques for indefinite and definite integration.
- Solve a wide range of real-world problems using calculus principles.
- Analyze the behavior of functions using calculus tools.
- Communicate mathematical solutions clearly and precisely.

Who This Course Is For

This Calculus 1 course is designed for students with a solid background in algebra and trigonometry. It is an intermediate-level course. You should be comfortable manipulating algebraic expressions, solving equations, and working with trigonometric functions and identities before you begin.

The course serves anyone seeking a comprehensive understanding of single-variable calculus. This includes university students, aspiring engineers, scientists, and anyone preparing for advanced quantitative studies.

Before You Start

- Proficiency in algebra.
- Familiarity with trigonometric functions and identities.
- Ability to solve algebraic equations.

How to Use This Guide

This study guide is designed to complement Professor Leonard's video lectures. Watch the video lessons first. Then, use this guide to review key concepts, work through examples, and complete practice problems.

The course is divided into six modules and 33 lessons. Pace yourself consistently. Many students find it helpful to dedicate specific time slots each week to their studies. Do not rush through the material.

Each module includes review sections and a glossary of terms. Use these tools to reinforce your understanding and to clarify any unfamiliar vocabulary. The final module contains a comprehensive assessment to test your knowledge.

CHAPTER 1

Section 1

This chapter lays the essential groundwork for Calculus 1. Before getting into the core concepts of limits and derivatives, we will review fundamental precalculus topics. A solid understanding of lines, functions, and trigonometry is crucial for success in calculus. We then introduce the concept of a limit, which is the cornerstone of calculus. You will learn how to evaluate limits using various techniques and understand what it means for a limit to exist or not exist. Finally, we connect limits to the idea of continuity and use them to define the slope of a curve and instantaneous rates of change, paving the way for the study of derivatives.

Foundations: Lines, Functions, and Trigonometry Review

Covers: Calculus 1 Lecture 0.1: Lines, Angle of Inclination, and the Distance Formula · Calculus 1 Lecture 0.2: Introduction to Functions · Calculus 1 Lecture 0.3: Review of Trigonometry and Graphing Trigonometric Functions · Calculus 1 Lecture 0.4: Combining and Composition of Functions

This initial section reviews essential precalculus concepts that form the bedrock of calculus. A strong command of these topics will greatly simplify your understanding of calculus. We begin with the properties of lines.

Lines, Angle of Inclination, and the Distance Formula

A straight line is defined by its slope, which measures its steepness. Given two points (x_1, y_1) and (x_2, y_2) , the slope m is calculated as $m = (y_2 - y_1) / (x_2 - x_1)$. The equation of a line can be expressed in point-slope form, $y - y_1 = m(x - x_1)$, or slope-intercept form, $y = mx + b$, where b is the y-intercept. Parallel lines have identical slopes, while perpendicular lines have slopes that are negative reciprocals of each other (i.e., $m_1 * m_2 = -1$).

The angle of inclination, θ , is the angle a line makes with the positive x-axis. It is related to the slope by $\tan(\theta) = m$. This angle is typically measured in radians and lies in the interval $[0, \pi)$. For example, a line with slope $m = 1$ has an angle of inclination of $\pi/4$ radians. The distance between two points (x_1, y_1) and (x_2, y_2) in a plane is found using the distance formula, derived from the Pythagorean theorem: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Introduction to Functions

A function is a rule that assigns to each input value in its domain exactly one output value in its range. We often write $y = f(x)$, where x is the independent variable (input) and y is the dependent variable (output). The domain is the set of all permissible input values, and the range is the set of all possible output values. The vertical line test helps identify if a graph represents a function: if any vertical line intersects the graph at more than one point, it is not a function.

Common types of functions include linear, quadratic, polynomial, rational, exponential, logarithmic, and trigonometric functions. Understanding their basic shapes, domains, and ranges is vital. For example, the

domain of $f(x) = 1/x$ excludes $x = 0$, and its range excludes $y = 0$. Polynomials generally have a domain of all real numbers.

Review of Trigonometry and Graphing Trigonometric Functions

Trigonometry is fundamental to calculus, especially when dealing with oscillatory behavior and rates of change involving angles. Recall the unit circle definitions for sine, cosine, and tangent. For an angle θ in standard position, $\cos(\theta)$ is the x-coordinate and $\sin(\theta)$ is the y-coordinate of the point where the terminal side intersects the unit circle. $\tan(\theta) = \sin(\theta) / \cos(\theta)$.

Key trigonometric identities, such as $\sin^2(\theta) + \cos^2(\theta) = 1$, are frequently used. Graphing trigonometric functions involves understanding their amplitude, period, and phase shifts. For instance, $y = A \sin(Bx + C) + D$ has amplitude $|A|$, period $2\pi/|B|$, and a phase shift determined by C . A solid recall of these functions' graphs and properties will be invaluable.

Combining and Composition of Functions

Functions can be combined using arithmetic operations. If f and g are functions, then:

- Sum: $(f + g)(x) = f(x) + g(x)$
- Difference: $(f - g)(x) = f(x) - g(x)$
- Product: $(fg)(x) = f(x)g(x)$
- Quotient: $(f / g)(x) = f(x) / g(x)$, provided $g(x) \neq 0$

The domain of these combined functions is the intersection of the domains of f and g , with an additional restriction for quotients that $g(x)$ cannot be zero.

Function composition involves applying one function to the result of another. The composition $(f \circ g)(x)$ is defined as $f(g(x))$. For $(f \circ g)(x)$ to be defined, the output of $g(x)$ must be within the domain of $f(x)$.

Example: Let $f(x) = x^2$ and $g(x) = x - 3$.

- $(f + g)(x) = x^2 + x - 3$
- $(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2 = x^2 - 6x + 9$
- $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$

Notice that $(f \circ g)(x)$ is generally not equal to $(g \circ f)(x)$. Pay close attention to the domain of the inner function and how its range affects the domain of the outer function during composition.

Introduction to Limits

Covers: Calculus 1 Lecture 1.1: An Introduction to Limits · Calculus 1 Lecture 1.2: Properties of Limits. Techniques of Limit Computation

The concept of a limit is the foundational idea upon which all of calculus is built. It allows us to analyze the behavior of functions at specific points or as they extend towards infinity, even when the function itself might not be defined at that point.

An Introduction to Limits

Intuitively, the limit of a function $f(x)$ as x approaches a number c is the value that $f(x)$ gets arbitrarily close to as x gets arbitrarily close to c (but not necessarily equal to c). We write this as $\lim_{x \rightarrow c} f(x) = L$.

To determine if a limit L exists, we must consider the behavior of $f(x)$ as x approaches c from both the left side (values less than c) and the right side (values greater than c). These are called one-sided limits:

- Left-hand limit: $\lim_{x \rightarrow c^-} f(x)$
- Right-hand limit: $\lim_{x \rightarrow c^+} f(x)$

For the two-sided limit $\lim_{x \rightarrow c} f(x)$ to exist and be equal to L , the left-hand limit and the right-hand limit must both exist and be equal to L . If $\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x)$, or if either one-sided limit does not exist (e.g., approaches infinity), then the two-sided limit $\lim_{x \rightarrow c} f(x)$ does not exist.

Limits can be estimated from graphs or tables of values. For instance, if $f(x)$ approaches 5 as x gets closer to 2 from both sides, then $\lim_{x \rightarrow 2} f(x) = 5$. It does not matter what $f(2)$ actually is, or even if $f(2)$ is defined.

Properties of Limits. Techniques of Limit Computation

Calculating limits directly from the definition can be cumbersome. Fortunately, a set of limit properties, often called limit laws, simplify the process. Assuming $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist, and k is a constant:

- $\lim_{x \rightarrow c} [f(x) + g(x)] = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$ (Sum Rule)
- $\lim_{x \rightarrow c} [f(x) - g(x)] = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x)$ (Difference Rule)
- $\lim_{x \rightarrow c} [k * f(x)] = k * \lim_{x \rightarrow c} f(x)$ (Constant Multiple Rule)
- $\lim_{x \rightarrow c} [f(x) * g(x)] = \lim_{x \rightarrow c} f(x) * \lim_{x \rightarrow c} g(x)$ (Product Rule)
- $\lim_{x \rightarrow c} [f(x) / g(x)] = \lim_{x \rightarrow c} f(x) / \lim_{x \rightarrow c} g(x)$, provided $\lim_{x \rightarrow c} g(x) \neq 0$ (Quotient Rule)
- $\lim_{x \rightarrow c} [f(x)]^n = [\lim_{x \rightarrow c} f(x)]^n$ (Power Rule)

For polynomial and rational functions, direct substitution is often the first technique to try. If $f(x)$ is a polynomial or a rational function (where the denominator is not zero at $x = c$), then $\lim_{x \rightarrow c} f(x) = f(c)$.

However, direct substitution often leads to indeterminate forms like $0/0$. When this occurs, algebraic manipulation is required to simplify the expression before re-evaluating the limit.

Techniques for Indeterminate Forms (0/0):

1. **Factoring and Canceling:** If the numerator and denominator share a common factor, factor both and cancel the common term. This removes the discontinuity.

$$\begin{aligned} \lim_{x \rightarrow 2} (x^2 - 4) / (x - 2) \\ = \lim_{x \rightarrow 2} (x - 2)(x + 2) / (x - 2) \end{aligned}$$

$$= \lim_{x \rightarrow 2} (x + 2)$$

$$= 2 + 2 = 4$$

2. Rationalizing Expressions: If the expression involves square roots, multiply the numerator and denominator by the conjugate of the term with the square root.

$$\begin{aligned} & \lim_{x \rightarrow 0} (\sqrt{x+1} - 1) / x \\ &= \lim_{x \rightarrow 0} ((\sqrt{x+1} - 1) / x) * ((\sqrt{x+1} + 1) / (\sqrt{x+1} + 1)) \\ &= \lim_{x \rightarrow 0} ((x+1) - 1) / (x * (\sqrt{x+1} + 1)) \\ &= \lim_{x \rightarrow 0} x / (x * (\sqrt{x+1} + 1)) \\ &= \lim_{x \rightarrow 0} 1 / (\sqrt{x+1} + 1) \\ &= 1 / (\sqrt{0+1} + 1) = 1 / (1 + 1) = 1/2 \end{aligned}$$

3. Finding a Common Denominator: For complex fractions, combine terms in the numerator or denominator to simplify.

$$\begin{aligned} & \lim_{x \rightarrow 0} ((1 / (x+3)) - (1 / 3)) / x \\ &= \lim_{x \rightarrow 0} ((3 - (x+3)) / (3(x+3))) / x \\ &= \lim_{x \rightarrow 0} (-x / (3(x+3))) / x \\ &= \lim_{x \rightarrow 0} -x / (3x(x+3)) \\ &= \lim_{x \rightarrow 0} -1 / (3(x+3)) \\ &= -1 / (3(0+3)) = -1/9 \end{aligned}$$

These algebraic techniques are crucial for evaluating limits that cannot be solved by direct substitution. Practice is key to mastering them.

Continuity and Rates of Change

Covers: Calculus 1 Lecture 1.4: Continuity of Functions · Calculus 1 Lecture 1.5: Slope of a Curve, Velocity, and Rates of Change

With a firm grasp of limits, we can now define continuity and explore its implications. We will also see how limits provide the mathematical framework for understanding the slope of a curve and instantaneous rates of change, which are central to differential calculus.

Continuity of Functions

A function $f(x)$ is said to be continuous at a point $x = c$ if its graph has no breaks, jumps, or holes at that point. More formally, continuity at $x = c$ requires three conditions to be met:

1. $f(c)$ is defined (the point exists).
2. $\lim_{x \rightarrow c} f(x)$ exists (the limit approaches a specific value from both sides).
3. $\lim_{x \rightarrow c} f(x) = f(c)$ (the limit value equals the function value at that point).

If any of these conditions fail, the function is discontinuous at $x = c$.

There are several types of discontinuities:

- **Removable Discontinuity (Hole):** Occurs when $\lim_{x \rightarrow c} f(x)$ exists, but $f(c)$ is either undefined or $f(c) \neq \lim_{x \rightarrow c} f(x)$. This can often be "removed" by redefining the function at that

single point. For example, $f(x) = (x^2 - 4) / (x - 2)$ has a removable discontinuity at $x = 2$.

- **Jump Discontinuity:** Occurs when the left-hand limit and the right-hand limit at $x = c$ both exist but are not equal. This is common in piecewise functions.
- **Infinite Discontinuity (Vertical Asymptote):** Occurs when $\lim_{x \rightarrow c} f(x)$ approaches positive or negative infinity. This means the function grows without bound as x approaches c . For example, $f(x) = 1/x$ has an infinite discontinuity at $x = 0$.

A function is continuous on an interval if it is continuous at every point within that interval. Polynomials, for example, are continuous everywhere. Rational functions are continuous on their domains.

The **Intermediate Value Theorem (IVT)** is an important consequence of continuity. It states that if f is continuous on a closed interval $[a, b]$, and N is any number between $f(a)$ and $f(b)$, then there exists at least one number c in (a, b) such that $f(c) = N$. This theorem guarantees the existence of a solution within an interval, though it does not provide a method to find it.

Slope of a Curve, Velocity, and Rates of Change

One of the most powerful applications of limits in calculus is defining the instantaneous rate of change. Consider a curve $y = f(x)$. We want to find the slope of the tangent line to the curve at a specific point $P(a, f(a))$.

We begin by considering a secant line connecting $P(a, f(a))$ and another point $Q(x, f(x))$ on the curve. The slope of this secant line is given by:

$$m_{\text{sec}} = (f(x) - f(a)) / (x - a)$$

As Q approaches P (meaning x approaches a), the secant line approaches the tangent line. Therefore, the slope of the tangent line, which represents the instantaneous rate of change of $f(x)$ at $x = a$, is defined as the limit of the secant slopes:

$$m_{\text{tan}} = \lim_{x \rightarrow a} (f(x) - f(a)) / (x - a)$$

An alternative, and often more convenient, form uses h as the distance between x and a , so $x = a + h$. As x approaches a , h approaches 0 . Substituting $x = a + h$ into the formula gives:

$$m_{\text{tan}} = \lim_{h \rightarrow 0} (f(a + h) - f(a)) / h$$

This limit is called the **difference quotient**. It is the fundamental definition of the derivative, which we will explore in detail in later chapters.

This concept has direct applications in physics and other sciences. If $f(t)$ represents the position of an object at time t , then the slope of the secant line $(f(t_2) - f(t_1)) / (t_2 - t_1)$ represents the average velocity over the time interval $[t_1, t_2]$. Taking the limit as t_2 approaches t_1 (or h approaches 0) yields the instantaneous velocity at time t_1 :

$$\text{Instantaneous Velocity} = \lim_{h \rightarrow 0} (f(t_1 + h) - f(t_1)) / h$$

This demonstrates how limits allow us to move from average rates of change over an interval to precise instantaneous rates of change at a single point, a critical step in understanding dynamic systems.

KEY TAKEAWAYS

- A strong grasp of algebra, functions, and trigonometry is essential for calculus.
- A function assigns exactly one output for each input.
- The limit of a function describes the value a function approaches as its input approaches a certain point.
- One-sided limits must be equal for a two-sided limit to exist.
- Continuity at a point requires the function to be defined, the limit to exist, and these two values to be equal.
- The slope of a tangent line, representing instantaneous rate of change, is defined using a limit.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. What are the three conditions for a function $f(x)$ to be continuous at a point $x = c$?
2. Explain the difference between the slope of a secant line and the slope of a tangent line.
3. When does the limit of a function $f(x)$ as x approaches c not exist?
4. Given $f(x) = x^2 + 1$ and $g(x) = 2x - 3$, find $(f \circ g)(x)$ and state its domain.
5. What is the angle of inclination of a line with a slope of $m = 1$?

CHAPTER 2

The Derivative: Rules and Applications

Chapter 2 introduces one of the two fundamental concepts of calculus: the derivative. You will learn what a derivative represents both algebraically and geometrically. This chapter provides the essential rules for calculating derivatives efficiently, moving beyond the limit definition to practical techniques.

Mastering these differentiation techniques is crucial. The chapter concludes by exploring various real-world applications of the derivative, including finding tangent lines, understanding rates of change, and solving related rates problems. A solid grasp of these concepts forms the foundation for further study in calculus.

Understanding and Calculating Basic Derivatives

Covers: Calculus 1 Lecture 2.1: Introduction to the Derivative of a Function · Calculus 1 Lecture 2.2: Techniques of Differentiation (Finding Derivatives of Functions Easily)

The derivative is a measure of how a function changes as its input changes. It represents the instantaneous rate of change of a function. Geometrically, the derivative at a point gives the slope of the tangent line to the function's graph at that point.

We define the derivative of a function $f(x)$ using a limit:

$$f'(x) = \lim_{h \rightarrow 0} [f(x+h) - f(x)] / h$$

This is the formal definition. Common notations for the derivative include $f'(x)$, dy/dx , and $d/dx [f(x)]$. A function is differentiable at a point if this limit exists. Derivatives do not exist at points of discontinuity, sharp corners (cusps), or vertical tangent lines.

While the limit definition is fundamental, it is often tedious to use. Fortunately, a set of rules simplifies the process of finding derivatives. These basic differentiation techniques are your first tools:

The Constant Rule

If c is a constant, then the derivative of c with respect to x is zero.

$$d/dx [c] = 0$$

For example, if $f(x) = 5$, then $f'(x) = 0$.

The Power Rule

If n is any real number, then the derivative of x^n with respect to x is n times x raised to the power of $n-1$.

$$d/dx [x^n] = n * x^{(n-1)}$$

Examples:

- If $f(x) = x^3$, then $f'(x) = 3x^2$.
- If $g(x) = x^{(1/2)}$ (which is the square root of x), then $g'(x) = (1/2)x^{(-1/2)}$.
- If $h(x) = 1/x = x^{(-1)}$, then $h'(x) = -1x^{(-2)} = -1/x^2$.

The Constant Multiple Rule

If c is a constant and $f(x)$ is a differentiable function, then the derivative of c times $f(x)$ is c times the derivative of $f(x)$.

$$d/dx [c * f(x)] = c * f'(x)$$

For instance, if $f(x) = 4x^5$, then $f'(x) = 4 * (5x^4) = 20x^4$.

The Sum and Difference Rules

If $f(x)$ and $g(x)$ are differentiable functions, then the derivative of their sum or difference is the sum or difference of their derivatives.

$$d/dx [f(x) +/- g(x)] = f'(x) +/- g'(x)$$

Example: Find the derivative of $y = 3x^4 + 2x - 7$.

- Apply the rules term by term:
- $d/dx [3x^4] = 3 * 4x^3 = 12x^3$
- $d/dx [2x] = 2 * 1x^0 = 2$
- $d/dx [-7] = 0$
- So, $dy/dx = 12x^3 + 2$.

Practice combining these rules. Remember to rewrite expressions with radicals or fractions as powers of x before applying the power rule. A common mistake is forgetting to apply the power rule correctly when n is negative or a fraction.

Mastering Advanced Differentiation Rules

Covers: Calculus 1 Lecture 2.3: The Product and Quotient Rules for Derivatives of Functions · Calculus 1 Lecture 2.5: Finding Derivatives of Trigonometric Functions · Calculus 1 Lecture 2.6: Discussion of the Chain Rule for Derivatives of Functions

As functions become more complex, we need additional rules beyond the basics. The product rule, quotient rule, and chain rule are essential for differentiating combinations of functions, while specific rules apply to trigonometric functions.

The Product Rule

When differentiating a product of two functions, $f(x)g(x)$, you cannot simply differentiate each function separately. The product rule states:

$$\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Often remembered as 'first derivative of the second plus second derivative of the first', or 'low d high minus high d low over low squared'. A common mnemonic for the product rule is "f prime g plus f g prime".

Example: Find the derivative of $y = (x^2 + 1)(3x - 2)$.

- Let $f(x) = x^2 + 1$, so $f'(x) = 2x$.
- Let $g(x) = 3x - 2$, so $g'(x) = 3$.
- $\frac{dy}{dx} = (2x)(3x - 2) + (x^2 + 1)(3)$
- $\frac{dy}{dx} = 6x^2 - 4x + 3x^2 + 3$
- $\frac{dy}{dx} = 9x^2 - 4x + 3$

The Quotient Rule

For a quotient of two functions, $f(x)/g(x)$, the rule is:

$$\frac{d}{dx} [f(x)/g(x)] = [f'(x)g(x) - f(x)g'(x)] / [g(x)]^2$$

This is often remembered as "low d high minus high d low, over low squared". Be careful with the order of subtraction in the numerator.

Example: Find the derivative of $y = (x^2) / (x - 3)$.

- Let $f(x) = x^2$, so $f'(x) = 2x$.
- Let $g(x) = x - 3$, so $g'(x) = 1$.
- $\frac{dy}{dx} = [(2x)(x - 3) - (x^2)(1)] / (x - 3)^2$
- $\frac{dy}{dx} = [2x^2 - 6x - x^2] / (x - 3)^2$
- $\frac{dy}{dx} = [x^2 - 6x] / (x - 3)^2$

Derivatives of Trigonometric Functions

You must know the derivatives of the six basic trigonometric functions:

- $\frac{d}{dx} [\sin(x)] = \cos(x)$
- $\frac{d}{dx} [\cos(x)] = -\sin(x)$
- $\frac{d}{dx} [\tan(x)] = \sec^2(x)$
- $\frac{d}{dx} [\cot(x)] = -\csc^2(x)$
- $\frac{d}{dx} [\sec(x)] = \sec(x)\tan(x)$

- $\frac{d}{dx} [\csc(x)] = -\csc(x) \cot(x)$

These can be derived using the limit definition or the quotient rule (e.g., $\tan(x) = \sin(x)/\cos(x)$).

The Chain Rule

The chain rule is used for differentiating composite functions, $f(g(x))$. It states that the derivative of $f(g(x))$ is the derivative of the 'outer' function evaluated at the 'inner' function, multiplied by the derivative of the 'inner' function.

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) * g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then $dy/dx = (dy/du) * (du/dx)$.

Example: Find the derivative of $y = (x^3 + 2x)^5$.

- Here, the outer function is u^5 and the inner function is $u = x^3 + 2x$.
- The derivative of the outer function is $5u^4$.
- The derivative of the inner function is $3x^2 + 2$.
- $dy/dx = 5(x^3 + 2x)^4 * (3x^2 + 2)$

Another example: Find the derivative of $y = \sin(4x)$.

- Outer function: $\sin(u)$, inner function: $u = 4x$.
- Derivative of outer: $\cos(u)$. Derivative of inner: 4 .
- $dy/dx = \cos(4x) * 4 = 4\cos(4x)$

Many functions require combinations of these rules. For example, $d/dx [\sin^2(x)]$ requires the chain rule (outer: u^2 , inner: $\sin(x)$) and the derivative of $\sin(x)$. This yields $2\sin(x)\cos(x)$.

Applications of the Derivative: Implicit and Related Rates

Covers: Calculus 1 Lecture 2.4: Applications of the Derivative · Calculus 1 Lecture 2.7: Implicit Differentiation · Calculus 1 Lecture 2.8: Related Rates

Derivatives are powerful tools for solving problems involving rates of change, optimization, and geometric properties of curves. Here, we explore some fundamental applications.

Tangent Lines and Rates of Change

The most direct application of the derivative $f'(x)$ is finding the slope of the tangent line to the graph of $f(x)$ at a specific point (x_0, y_0) . Once you have the slope $m = f'(x_0)$, you can use the point-slope form $y - y_0 = m(x - x_0)$ to write the equation of the tangent line.

Beyond geometry, the derivative represents a rate of change. If $s(t)$ is a position function, then $s'(t)$ is the instantaneous velocity, and $s''(t)$ (the second derivative) is the instantaneous acceleration. In economics, the derivative of a cost function is marginal cost, representing the cost to produce one more unit.

Implicit Differentiation

Sometimes, a function y is not explicitly defined as $y = f(x)$, but rather implicitly through an equation relating x and y , such as $x^2 + y^2 = 25$. To find dy/dx in such cases, we use implicit differentiation.

The process involves differentiating both sides of the equation with respect to x . When differentiating terms involving y , remember to apply the chain rule. Treat y as an unknown function of x , so $d/dx [y^n] = ny^{(n-1)} * (dy/dx)$.

Example: Find dy/dx for $x^2 + y^2 = 25$.

1. Differentiate both sides with respect to x : $d/dx [x^2] + d/dx [y^2] = d/dx [25]$
2. Apply differentiation rules: $2x + 2y * (dy/dx) = 0$
3. Isolate dy/dx : $2y * (dy/dx) = -2x$ $dy/dx = -2x / (2y)$ $dy/dx = -x / y$

Implicit differentiation allows us to find slopes of tangent lines for curves that are not functions, such as circles.

Related Rates

Related rates problems involve finding the rate at which one quantity is changing by relating it to other quantities whose rates of change are known. These problems typically involve implicit differentiation with respect to time t .

Steps to solve related rates problems:

1. **Draw a diagram:** Sketch the situation and label all known and unknown quantities and rates.
2. **Identify variables and rates:** List all variables and their given rates of change. Identify the rate you need to find.
3. **Find an equation:** Write an equation that relates the quantities involved. This often comes from geometry (Pythagorean theorem, area, volume formulas) or other physical principles.
4. **Differentiate implicitly:** Differentiate both sides of the equation with respect to time t . Remember to use the chain rule for any variable that is a function of t .
5. **Substitute and solve:** Substitute all known values (quantities and rates at the specific instant in question) into the differentiated equation and solve for the unknown rate.

Example: A ladder 10 feet long leans against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom is 6 feet from the wall?

1. Let x be the distance from the wall to the bottom of the ladder, and y be the height of the top of the ladder on the wall. The ladder length is 10 ft.
2. Knowns: $dx/dt = 1$ ft/s. We want to find dy/dt when $x = 6$ ft.
3. Equation: By the Pythagorean theorem, $x^2 + y^2 = 10^2 = 100$.
4. Differentiate with respect to t : $d/dt [x^2 + y^2] = d/dt [100]$ $2x * (dx/dt) + 2y * (dy/dt) = 0$
5. When $x = 6$, we find y : $6^2 + y^2 = 100 \Rightarrow 36 + y^2 = 100 \Rightarrow y^2 = 64 \Rightarrow y = 8$.

6. Substitute values: $2(6)(1) + 2(8)(dy/dt) = 0$ $12 + 16(dy/dt) = 0$ $16(dy/dt) = -12$ $dy/dt = -12 / 16 = -3/4$ ft/s

The negative sign indicates that the top of the ladder is moving downwards.

KEY TAKEAWAYS

- The derivative measures the instantaneous rate of change of a function and represents the slope of the tangent line to its graph.
- Basic differentiation rules include the constant rule, power rule, constant multiple rule, and sum/difference rules.
- The product rule and quotient rule are used to differentiate functions that are products or quotients of other functions.
- Specific rules exist for differentiating trigonometric functions, which must be memorized or derived.
- The chain rule is crucial for differentiating composite functions, requiring the product of the outer derivative and the inner derivative.
- Implicit differentiation allows us to find derivatives for equations where y is not explicitly defined as a function of x .
- Related rates problems use implicit differentiation with respect to time to solve for unknown rates of change.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. What does the derivative of a function represent both geometrically and as a rate of change?
2. Explain when you would use the Product Rule versus the Chain Rule, and provide a simple example for each.
3. List the derivatives of the six basic trigonometric functions.
4. Describe the general procedure for solving a related rates problem.
5. Find the derivative of $f(x) = (3x^2 - 5)^4$.

CHAPTER 3

Analyzing Functions and Introduction to Integration

This chapter expands your understanding of derivatives, using them to analyze the behavior of functions in detail. You will learn how to determine where a function is increasing or decreasing, identify its concavity, and locate local extrema. We will also cover important theorems like Rolle's Theorem and the Mean Value Theorem, and explore limits as functions approach infinity.

The skills you gain will culminate in sketching accurate graphs of functions and solving real-world optimization problems. Finally, we introduce the concept of the indefinite integral, laying the groundwork for the second major branch of calculus.

Analyzing Function Behavior with Derivatives

Covers: Calculus 1 Lecture 3.1: Increasing/Decreasing and Concavity of Functions · Calculus 1 Lecture 3.3: The First Derivative Test for Increasing and Decreasing · Calculus 1 Lecture 3.4: The Second Derivative Test for Concavity of Functions

This section focuses on using the first and second derivatives to understand the shape and behavior of a function's graph. We can determine where a function is increasing or decreasing, where its graph is concave up or concave down, and identify local maximum and minimum points. These analytical tools are crucial for sketching accurate graphs and solving applied problems.

Increasing and Decreasing Functions

A function $f(x)$ is **increasing** on an interval if, as x increases, $f(x)$ also increases. Graphically, the function rises from left to right. Conversely, $f(x)$ is **decreasing** on an interval if, as x increases, $f(x)$ decreases. Graphically, the function falls from left to right.

The first derivative provides a direct way to test for increasing or decreasing behavior:

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.
- If $f'(x) = 0$ at a point, that point is a critical number, potentially indicating a local extremum.

The First Derivative Test

The First Derivative Test helps identify local maxima and minima. Critical numbers are points where $f'(x) = 0$ or $f'(x)$ is undefined. These are the only places where a function can change from increasing to decreasing or vice versa.

To apply the First Derivative Test:

1. Find the first derivative, $f'(x)$.
2. Find the critical numbers by setting $f'(x) = 0$ and finding where $f'(x)$ is undefined.

3. Create a number line and mark the critical numbers.
4. Test a value in each interval created by the critical numbers to determine the sign of $f'(x)$.
5. If $f'(x)$ changes from positive to negative at a critical number c , then $f(c)$ is a local maximum.
6. If $f'(x)$ changes from negative to positive at a critical number c , then $f(c)$ is a local minimum.
7. If $f'(x)$ does not change sign, there is no local extremum at c .

Example: Find intervals where $f(x) = x^3 - 3x^2 + 2$ is increasing or decreasing, and identify local extrema.

1. $f'(x) = 3x^2 - 6x$
2. Set $f'(x) = 0$: $3x(x - 2) = 0$. Critical numbers are $x = 0$ and $x = 2$.
3. Test intervals: $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$.
 - * For $x = -1$, $f'(-1) = 9 > 0$. Increasing.
 - * For $x = 1$, $f'(1) = -3 < 0$. Decreasing.
 - * For $x = 3$, $f'(3) = 9 > 0$. Increasing.
4. $f(x)$ is increasing on $(-\infty, 0)$ and $(2, \infty)$.
5. $f(x)$ is decreasing on $(0, 2)$.
6. At $x = 0$, $f'(x)$ changes from positive to negative, so $f(0) = 2$ is a local maximum.
7. At $x = 2$, $f'(x)$ changes from negative to positive, so $f(2) = -2$ is a local minimum.

Concavity and Inflection Points

Concavity describes the direction of the curve's opening.

- A function is **concave up** on an interval if its graph lies above its tangent lines. It looks like a cup holding water.
- A function is **concave down** on an interval if its graph lies below its tangent lines. It looks like an inverted cup.

The second derivative determines concavity:

- If $f''(x) > 0$ on an interval, then $f(x)$ is concave up.
- If $f''(x) < 0$ on an interval, then $f(x)$ is concave down.

An **inflection point** is a point where the concavity of the function changes. This occurs where $f''(x) = 0$ or $f''(x)$ is undefined, provided $f'(x)$ changes sign at that point.

The Second Derivative Test

The Second Derivative Test can also identify local extrema, sometimes more easily than the First Derivative Test.

1. Find the first derivative, $f'(x)$, and find critical numbers where $f'(x) = 0$.
2. Find the second derivative, $f''(x)$.
3. Substitute each critical number c into $f''(x)$:
 - If $f''(c) > 0$, then $f(c)$ is a local minimum.
 - If $f''(c) < 0$, then $f(c)$ is a local maximum.

- If $f'(c) = 0$, the test is inconclusive. Use the First Derivative Test instead.

Example: Use the Second Derivative Test for $f(x) = x^3 - 3x^2 + 2$.

1. Critical numbers are $x = 0$ and $x = 2$.
2. $f''(x) = 6x - 6$.
3. For $x = 0$: $f''(0) = 6(0) - 6 = -6 < 0$. So, $f(0) = 2$ is a local maximum.
4. For $x = 2$: $f''(2) = 6(2) - 6 = 12 - 6 = 6 > 0$. So, $f(2) = -2$ is a local minimum.

Practice applying both tests. Understanding the relationship between a function and its derivatives is fundamental.

Important Theorems and Limits at Infinity

Covers: Calculus 1 Lecture 3.2: A BRIEF Discussion of Rolle's Theorem and Mean-Value Theorem. · Calculus 1 Lecture 3.5: Limits of Functions at Infinity

This section introduces two foundational theorems in differential calculus: Rolle's Theorem and the Mean Value Theorem. These theorems establish important relationships between the average rate of change and the instantaneous rate of change of a function. We will also extend our understanding of limits to examine the behavior of functions as their input values grow infinitely large or small.

Rolle's Theorem

Rolle's Theorem is a special case of the Mean Value Theorem. It states that if a function $f(x)$ satisfies three conditions on a closed interval $[a, b]$:

1. $f(x)$ is continuous on $[a, b]$.
2. $f(x)$ is differentiable on (a, b) .
3. $f(a) = f(b)$.

Then there exists at least one number c in (a, b) such that $f'(c) = 0$.

In simpler terms, if a continuous and differentiable function starts and ends at the same y -value over an interval, there must be at least one point within that interval where the tangent line is horizontal. This means the instantaneous rate of change is zero.

Example: Verify Rolle's Theorem for $f(x) = x^2 - 4x + 3$ on $[1, 3]$.

1. $f(x)$ is a polynomial, so it is continuous on $[1, 3]$ and differentiable on $(1, 3)$.
2. Check $f(a)$ and $f(b)$:
 - * $f(1) = 1^2 - 4(1) + 3 = 0$.
 - * $f(3) = 3^2 - 4(3) + 3 = 0$.
 - * Since $f(1) = f(3) = 0$, the third condition is met.
3. Find c such that $f'(c) = 0$:
 - * $f'(x) = 2x - 4$.
 - * Set $2x - 4 = 0$ to get $x = 2$.
 - * The value $c = 2$ is in the interval $(1, 3)$.
 - * Thus, Rolle's Theorem is verified.

The Mean Value Theorem (MVT)

The Mean Value Theorem is a more general and powerful statement. It states that if a function $f(x)$ satisfies two conditions on a closed interval $[a, b]$:

1. $f(x)$ is continuous on $[a, b]$.
2. $f(x)$ is differentiable on (a, b) .

Then there exists at least one number c in (a, b) such that: $f'(c) = (f(b) - f(a)) / (b - a)$

This theorem states that there must be at least one point c where the instantaneous rate of change ($f'(c)$) is equal to the average rate of change of the function over the entire interval $[a, b]$. Graphically, this means there is a point c where the tangent line is parallel to the secant line connecting the endpoints $(a, f(a))$ and $(b, f(b))$.

Example: Verify the MVT for $f(x) = x^2$ on $[0, 2]$.

1. $f(x)$ is a polynomial, so it is continuous on $[0, 2]$ and differentiable on $(0, 2)$.
2. Calculate the average rate of change:

$$* (f(2) - f(0)) / (2 - 0) = (2^2 - 0^2) / 2 = 4 / 2 = 2.$$
3. Find c such that $f'(c)$ equals the average rate of change:

$$* f'(x) = 2x.$$

$$* \text{Set } 2c = 2 \text{ to get } c = 1.$$

$$* \text{The value } c = 1 \text{ is in the interval } (0, 2).$$

$$* \text{Thus, the Mean Value Theorem is verified.}$$

Limits of Functions at Infinity

We have previously studied limits as x approaches a finite number. Now, we consider what happens to $f(x)$ as x approaches positive or negative infinity. These are called **limits at infinity**. They help us understand the end behavior of a function and identify horizontal asymptotes.

- If $\lim_{x \rightarrow \infty} f(x) = L$, then $y = L$ is a horizontal asymptote.
- If $\lim_{x \rightarrow -\infty} f(x) = L$, then $y = L$ is a horizontal asymptote.

To evaluate limits at infinity for rational functions (polynomial divided by polynomial), a common technique is to divide every term in the numerator and denominator by the highest power of x in the denominator.

Example: Evaluate $\lim_{x \rightarrow \infty} (3x^2 + 2x - 1) / (x^2 + 4x + 5)$.

Divide numerator and denominator by x^2 :

$$\lim_{x \rightarrow \infty} (3 + 2/x - 1/x^2) / (1 + 4/x + 5/x^2)$$

As x approaches infinity, terms like $2/x$, $1/x^2$, $4/x$, and $5/x^2$ all approach 0.
 So the limit becomes $(3 + 0 - 0) / (1 + 0 + 0) = 3 / 1 = 3$.

Therefore, $y = 3$ is a horizontal asymptote.

Remember that for polynomials, the limit as x approaches infinity is determined by the leading term. For rational functions, compare the degrees of the numerator and denominator.

Graphing and Optimization

Covers: Calculus 1 Lecture 3.6: How to Sketch Graphs of Functions · Calculus 1 Lecture 3.7: Optimization; Max/Min Application Problems

This section brings together many of the derivative concepts to achieve two key goals: sketching accurate graphs of functions and solving real-world optimization problems. Graphing a function systematically requires analyzing its domain, intercepts, asymptotes, and behavior using the first and second derivatives. Optimization problems involve finding the maximum or minimum value of a quantity under given constraints, which is a direct application of finding local extrema.

How to Sketch Graphs of Functions

Sketching a graph using calculus involves a systematic approach:

1. **Domain:** Identify the set of all possible x values for the function. Look for restrictions like division by zero or square roots of negative numbers.
2. **Intercepts:**
 - y -intercept: Set $x = 0$ and solve for y .
 - x -intercepts: Set $f(x) = 0$ and solve for x .
3. **Symmetry:** Check if the function is even ($f(-x) = f(x)$, symmetric about the y -axis) or odd ($f(-x) = -f(x)$, symmetric about the origin).
4. **Asymptotes:**
 - *Vertical asymptotes:* Occur where the denominator is zero and the numerator is non-zero (for rational functions).
 - *Horizontal asymptotes:* Determine $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.
 - *Slant (oblique) asymptotes:* Occur if the degree of the numerator is exactly one greater than the degree of the denominator (for rational functions, found by polynomial long division).
5. **First Derivative Analysis:**
 - Find $f'(x)$.
 - Find critical numbers (where $f'(x) = 0$ or $f'(x)$ is undefined).
 - Use the First Derivative Test to determine intervals of increasing/decreasing and locate local maxima/minima.
6. **Second Derivative Analysis:**
 - Find $f''(x)$.
 - Find potential inflection points (where $f''(x) = 0$ or $f''(x)$ is undefined).
 - Use the Second Derivative Test for concavity to determine intervals of concave up/down and confirm inflection points.

7. **Plot Points and Sketch:** Plot intercepts, critical points, inflection points, and any points used for testing intervals. Draw asymptotes. Connect the points smoothly, following the increasing/decreasing and concavity information.

This systematic process ensures you capture all significant features of the function's graph. Do not skip steps. Each piece of information builds upon the others.

Optimization; Max/Min Application Problems

Optimization problems involve finding the maximum or minimum value of a quantity. These are common in business, engineering, and science. The process typically involves:

1. **Understand the Problem:** Read carefully. Identify the quantity to be optimized (e.g., area, volume, cost, time).
2. **Draw a Diagram:** If applicable, sketch the situation and label variables.
3. **Formulate the Objective Function:** Write an equation for the quantity to be maximized or minimized. This function will usually have more than one variable initially.
4. **Identify Constraints:** Write equations relating the variables based on the problem's conditions. These constraints allow you to reduce the objective function to a single variable.
5. **Express Objective Function in One Variable:** Use the constraint equations to substitute and rewrite the objective function with only one independent variable.
6. **Find the Domain:** Determine the reasonable range of values for the independent variable based on the problem's context.
7. **Find Critical Numbers:** Differentiate the objective function with respect to its single variable. Set the derivative equal to zero and solve for critical numbers. Also consider where the derivative is undefined.
8. **Test Critical Numbers and Endpoints:** Use the First or Second Derivative Test to classify critical numbers as local maxima or minima. If the domain is a closed interval, also evaluate the objective function at the endpoints. The absolute maximum or minimum will be the largest or smallest of these values.
9. **State the Answer:** Clearly answer the original question, including units.

Example: A farmer has 1000 feet of fencing and wants to enclose a rectangular field bordering a straight river. No fence is needed along the river. What dimensions will maximize the area of the field?

1. Optimize: Maximize area.
2. Diagram: A rectangle with one side along the river. Let length parallel to river be L and width perpendicular to river be W .
3. Objective Function: Area $A = L * W$.
4. Constraint: $2W + L = 1000$ (total fencing).
5. One Variable: From constraint, $L = 1000 - 2W$. Substitute into Area: $A(W) = (1000 - 2W)W = 1000W - 2W^2$.
6. Domain: $W > 0$. Also, $1000 - 2W > 0$ implies $W < 500$. So, $0 < W < 500$.
7. Critical Numbers: $A'(W) = 1000 - 4W$. Set $A'(W) = 0$: $1000 - 4W = 0$ implies $4W = 1000$, so $W = 250$.
8. Test: $A''(W) = -4$. Since $A''(250) = -4 < 0$, this is a local maximum.

9. Answer: If $W = 250$ feet, then $L = 1000 - 2(250) = 1000 - 500 = 500$ feet.

The dimensions are 250 feet by 500 feet. The maximum area is $250 * 500 = 125,000$ square feet.

Careful setup and execution of these steps are key to solving optimization problems successfully.

An Introduction to the Indefinite Integral

Covers: Calculus 1 Lecture 4.1: An Introduction to the Indefinite Integral

This section introduces the second major branch of calculus: integration. While differentiation helps us find the rate of change of a function, integration helps us reverse this process. We will begin with the concept of the indefinite integral, also known as the antiderivative. This is a fundamental concept that will lead to calculating areas, volumes, and solving many other problems in calculus.

Antiderivatives

An **antiderivative** of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. In simpler terms, if you differentiate $F(x)$, you get $f(x)$.

For example, if $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative because the derivative of x^2 is $2x$. However, $F(x) = x^2 + 5$ is also an antiderivative, because the derivative of $x^2 + 5$ is also $2x$. In fact, $F(x) = x^2 + C$ for any constant C is an antiderivative of $f(x) = 2x$. This constant C is called the **constant of integration**.

The Indefinite Integral

The process of finding antiderivatives is called **integration**. The symbol for the indefinite integral is $\int$. The indefinite integral of $f(x)$ with respect to x is written as: $\int f(x) dx = F(x) + C$ where $F'(x) = f(x)$ and C is the constant of integration. The dx indicates that x is the variable of integration.

Basic Integration Rules

Many integration rules are simply the reverse of differentiation rules.

- Power Rule for Integration:** If $n \neq -1$, then $\int x^n dx = (x^{(n+1)}) / (n+1) + C$. Example: $\int x^3 dx = (x^{(3+1)}) / (3+1) + C = x^4 / 4 + C$. Check: The derivative of $x^4 / 4 + C$ is $4x^3 / 4 = x^3$.
- Constant Multiple Rule:** $\int k * f(x) dx = k * \int f(x) dx$. Example: $\int 5x^2 dx = 5 \int x^2 dx = 5 * (x^3 / 3) + C = (5/3)x^3 + C$.
- Sum/Difference Rule:** $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$. Example: $\int (x^2 + 2x) dx = \int x^2 dx + \int 2x dx = (x^3 / 3) + x^2 + C$.
- Integral of a Constant:** $\int k dx = kx + C$. Example: $\int 7 dx = 7x + C$.
- Integral of $1/x$:** $\int (1/x) dx = \ln|x| + C$. (Recall that the derivative of $\ln|x|$ is $1/x$).
- Integrals of Exponential Functions:** $\int e^x dx = e^x + C$. $\int a^x dx = a^x / \ln(a) + C$ (for $a > 0, a \neq 1$).
- Integrals of Trigonometric Functions:**

- $\int \cos(x) \, dx = \sin(x) + C$
- $\int \sin(x) \, dx = -\cos(x) + C$
- $\int \sec^2(x) \, dx = \tan(x) + C$
- $\int \csc^2(x) \, dx = -\cot(x) + C$
- $\int \sec(x)\tan(x) \, dx = \sec(x) + C$
- $\int \csc(x)\cot(x) \, dx = -\csc(x) + C$

Initial Value Problems

To find a *specific* antiderivative (i.e., to determine the value of C), we need an **initial condition** or **initial value**. This is a point (x_0, y_0) that the function passes through.

Example: Find $F(x)$ if $F'(x) = 3x^2 - 2x$ and $F(1) = 4$.

1. Find the general antiderivative:

$$F(x) = \int (3x^2 - 2x) \, dx = 3(x^3 / 3) - 2(x^2 / 2) + C = x^3 - x^2 + C.$$
2. Use the initial condition $F(1) = 4$:

$$4 = (1)^3 - (1)^2 + C$$

$$4 = 1 - 1 + C$$

$$4 = C.$$
3. The specific antiderivative is $F(x) = x^3 - x^2 + 4$.

Understanding antiderivatives is the first step towards understanding definite integrals and their applications. Practice these basic rules until they become second nature.

KEY TAKEAWAYS

- The first derivative indicates where a function is increasing or decreasing and helps identify local extrema.
- The second derivative determines the concavity of a function and helps locate inflection points.
- Rolle's Theorem and the Mean Value Theorem relate average and instantaneous rates of change.
- Limits at infinity describe a function's end behavior and reveal horizontal asymptotes.
- Systematic graphing involves analyzing domain, intercepts, asymptotes, and derivative information.
- Optimization problems use derivatives to find maximum or minimum values in real-world contexts.
- The indefinite integral is the reverse process of differentiation, finding a function given its derivative.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. How do the first and second derivatives help in sketching the graph of a function?
2. Explain the relationship between Rolle's Theorem and the Mean Value Theorem.
3. What is a horizontal asymptote, and how is it found using limits?
4. Outline the general steps for solving an optimization problem.
5. What is an antiderivative, and why is there a 'constant of integration' when finding indefinite integrals?

CHAPTER 4

Integration and Its Applications

Chapter 4 introduces the powerful concept of integration, a cornerstone of calculus. We begin by mastering integration by substitution, a vital technique for solving a wide range of integrals. The chapter then builds a foundational understanding of the definite integral, connecting it to the idea of finding the area under a curve through Riemann sums. This leads directly to the Fundamental Theorem of Calculus, which provides an elegant and efficient method for evaluating definite integrals. Finally, we explore practical applications, using definite integrals to calculate areas between curves and volumes of solids of revolution.

Integration by Substitution

Covers: Calculus 1 Lecture 4.2: Integration by Substitution (video, 1 hr 34 min)

Integration by substitution is a fundamental technique for finding antiderivatives. It is essentially the reverse of the chain rule for differentiation. When you differentiate a composite function, say $f(g(x))$, the chain rule gives $f'(g(x)) \cdot g'(x)$. Integration by substitution helps us reverse this process. We look for an integrand that has a function and its derivative (or a constant multiple of its derivative) present.

The core idea is to simplify the integral by introducing a new variable, typically u .

The Substitution Method Steps

1. **Choose u :** Select a part of the integrand to be u . Often, u is the "inside" function of a composite function, or the argument of a trigonometric or exponential function.
2. **Find du :** Differentiate u with respect to x to find du/dx , then solve for du . This du will include dx .
3. **Substitute:** Replace u and du into the integral. The goal is to transform the integral into a simpler form involving only u .
4. **Integrate:** Evaluate the new integral with respect to u .
5. **Substitute Back:** Replace u with its original expression in terms of x to get the final answer. Remember to add the constant of integration, C , for indefinite integrals.

Example: Indefinite Integral

Let's evaluate $\int x^2 (x^3 + 5)^4 dx$.

1. **Choose u :** Let $u = x^3 + 5$.
2. **Find du :** $du/dx = 3x^2$, so $du = 3x^2 dx$.
3. **Substitute:** We have $x^2 dx$ in the original integral. From $du = 3x^2 dx$, we can write $x^2 dx = (1/3) du$. The integral becomes $\int (u)^4 (1/3) du$.
4. **Integrate:** $(1/3) \int u^4 du = (1/3) (u^5/5) + C = u^5/15 + C$.
5. **Substitute Back:** Replace u with $x^3 + 5$. The result is $(x^3 + 5)^5 / 15 + C$.

Common Mistakes

- **Forgetting dx and du :** Always ensure dx is replaced by du in the substitution step.
- **Missing constants:** Do not forget the constant factor that often arises when solving for dx or du .
- **Not substituting back:** For indefinite integrals, the final answer must be in terms of the original variable.

Definite Integrals with Substitution

When performing substitution on a definite integral, you have two options:

1. **Change the limits:** Convert the original limits of integration (in terms of x) to new limits (in terms of u) using your u substitution. Then, integrate with respect to u and evaluate using the new u limits. Do not substitute x back in this case.
2. **Integrate first, then substitute back:** Find the indefinite integral in terms of x , then apply the original x limits.

Changing the limits is often more efficient.

Example: Definite Integral

Evaluate $\int_0^1 x \sqrt{x^2 + 1} \, dx$.

1. **Choose u :** Let $u = x^2 + 1$.
2. **Find du :** $\frac{du}{dx} = 2x$, so $du = 2x \, dx$. This means $x \, dx = \frac{1}{2} du$.
3. **Change Limits:** When $x = 0$, $u = 0^2 + 1 = 1$. When $x = 1$, $u = 1^2 + 1 = 2$. The integral becomes $\int_1^2 \sqrt{u} \cdot \frac{1}{2} du$.
4. **Integrate:** $\frac{1}{2} \int_1^2 u^{1/2} du = \frac{1}{2} \left[\frac{u^{3/2}}{3/2} \right]_1^2 = \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_1^2 = \frac{1}{3} \left[u^{3/2} \right]_1^2$.
5. **Evaluate:** $\frac{1}{3} (2^{3/2} - 1^{3/2}) = \frac{1}{3} (2\sqrt{2} - 1)$.

Practice identifying suitable u substitutions. Look for compositions of functions where the derivative of the inner function is present elsewhere in the integrand.

The Definite Integral and the Fundamental Theorem of Calculus

Covers: Calculus 1 Lecture 4.3: Area Under a Curve, Limit Approach, Riemann Sums (video, 2 hr 7 min) · Calculus 1 Lecture 4.4: The Evaluation of Definite Integrals (video, 31 min) · Calculus 1 Lecture 4.5: The Fundamental Theorem of Calculus (video, 2 hr 46 min)

The definite integral is a powerful tool for calculating the net accumulated change of a quantity, most commonly represented as the area under a curve. We first approach this concept using Riemann sums, which provide a foundational understanding.

Riemann Sums: Approximating Area

To find the area under the curve of a non-negative function $f(x)$ from $x=a$ to $x=b$, we can approximate it using rectangles.

1. **Divide the interval:** Split the interval $[a, b]$ into n subintervals of equal width, $\Delta x = (b - a)/n$.
2. **Choose sample points:** Within each subinterval, choose a sample point x_i^* . Common choices include the left endpoint, right endpoint, or midpoint of the subinterval.
3. **Form rectangles:** For each subinterval, construct a rectangle with width Δx and height $f(x_i^*)$. The area of each rectangle is $f(x_i^*) \Delta x$.
4. **Sum the areas:** The sum of the areas of these n rectangles is an approximation of the total area under the curve: $A \approx \sum_{i=1}^n f(x_i^*) \Delta x$.

As the number of rectangles n approaches infinity, and thus Δx approaches zero, this approximation becomes exact. This limit defines the definite integral.

The Definite Integral

The definite integral of $f(x)$ from a to b is denoted by $\int_a^b f(x) dx$ and is defined as:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Here, a is the lower limit of integration and b is the upper limit of integration. The result of a definite integral is a single number, representing the net signed area between the function's graph and the x -axis over the interval $[a, b]$. Areas above the x -axis are positive, and areas below are negative.

The Fundamental Theorem of Calculus (FTC)

The FTC is a cornerstone of calculus, establishing a profound connection between differentiation and integration. It has two parts.

FTC Part 1 (Derivative of an Integral)

If f is continuous on $[a, b]$, then the function $g(x)$ defined by:

$$g(x) = \int_a^x f(t) dt$$

is continuous on $[a, b]$ and differentiable on (a, b) , and its derivative is $g'(x) = f(x)$. This means that differentiation "undoes" integration.

FTC Part 2 (Evaluation of Definite Integrals)

If f is continuous on $[a, b]$ and F is any antiderivative of f (meaning $F'(x) = f(x)$), then:

$$\int_a^b f(x) dx = F(b) - F(a)$$

This part provides a powerful method for evaluating definite integrals without resorting to the tedious limit of Riemann sums. We simply find an antiderivative of the integrand and evaluate it at the upper and lower limits, then subtract.

Example: Evaluating a Definite Integral using FTC Part 2

Evaluate $\int_1^3 (x^2 + 2x) dx$.

- Find an antiderivative:** An antiderivative of $f(x) = x^2 + 2x$ is $F(x) = x^3/3 + x^2$.
- Evaluate at limits:** $F(3) = (3^3/3) + 3^2 = 27/3 + 9 = 9 + 9 = 18$. $F(1) = (1^3/3) + 1^2 = 1/3 + 1 = 4/3$.
- Subtract:** $\int_1^3 (x^2 + 2x) dx = F(3) - F(1) = 18 - 4/3 = 54/3 - 4/3 = 50/3$.

This result, $50/3$, represents the net area under the curve $y = x^2 + 2x$ from $x=1$ to $x=3$. The FTC makes definite integral evaluation straightforward once an antiderivative is found.

Applications of Definite Integrals

Covers: Calculus 1 Lecture 5.1: Finding Area Between Two Curves (video, 1 hr 34 min) · Calculus 1 Lecture 5.2: Volume of Solids By Disks and Washers Method (video, 2 hr 48 min)

Definite integrals are versatile tools for calculating various quantities beyond just area under a single curve. They are used to find areas between curves and volumes of solids of revolution.

Area Between Two Curves

To find the area enclosed by two continuous functions, $f(x)$ and $g(x)$, over an interval $[a, b]$, where $f(x) \geq g(x)$ for all x in the interval, the area A is given by:

$$A = \int_a^b [f(x) - g(x)] dx$$

Here, $f(x)$ is the "upper" function and $g(x)$ is the "lower" function.

Steps for Finding Area Between Curves

- Sketch the graphs:** Identify which function is above the other and find intersection points.
- Find intersection points:** Set $f(x) = g(x)$ to determine limits of integration, a and b .
- Set up the integral:** Formulate $\int_a^b [(\text{upper function}) - (\text{lower function})] dx$.
- Evaluate the integral:** Use the Fundamental Theorem of Calculus.

Example: Area Between Two Curves

Find the area bounded by $y = x^2$ and $y = x + 2$.

- Intersection points:** $x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0$. So, $x = -1$ and $x = 2$. These are the limits.
- Upper/Lower functions:** Between $x=-1$ and $x=2$, the line $y=x+2$ is above the parabola $y=x^2$.
- Set up integral:** $A = \int_{-1}^2 [(x+2) - x^2] dx$
- Evaluate:** $A = \int_{-1}^2 (-x^2 + x + 2) dx$ $A = [-x^3/3 + x^2/2 + 2x]_{-1}^2$ $A = [-(2^3)/3 + (2^2)/2 + 2(2)] - [-(-1)^3/3 + (-1)^2/2 + 2(-1)]$ $A = [-8/3 + 2 + 4] - [1/3 + 1/2 - 2]$ $A = [10/3] -$

$[-7/6]$ $A = 10/3 + 7/6 = 20/6 + 7/6 = 27/6 = 9/2$. The area is $9/2$ square units.

Volumes of Solids By Disks and Washers Method

This method finds the volume of a solid of revolution, formed by rotating a two-dimensional region around an axis.

Disk Method

Used when the region being revolved is flush against the axis of revolution. The volume of a single disk is $\pi (\text{radius})^2 (\text{thickness})$. For $y=f(x)$ revolved around the x-axis from $x=a$ to $x=b$:

$$V = \int_a^b \pi [f(x)]^2 dx$$

Here, $f(x)$ is the radius $R(x)$.

Washer Method

Used when there is a gap between the region and the axis of revolution, forming a solid with a hole. The volume of a single washer is $\pi [(\text{outer radius})^2 - (\text{inner radius})^2] (\text{thickness})$. For a region bounded by $y=f(x)$ (outer) and $y=g(x)$ (inner) revolved around the x-axis from $x=a$ to $x=b$:

$$V = \int_a^b \pi ([f(x)]^2 - [g(x)]^2) dx$$

Here, $f(x)$ is the outer radius $R(x)$ and $g(x)$ is the inner radius $r(x)$.

Key Considerations

- **Axis of Revolution:** If revolving around the y-axis, functions must be $x=f(y)$ and the integral with respect to dy .
- **Draw a Diagram:** Always sketch the region and axis to correctly identify radii and limits.

Example: Volume by Disk Method

Find the volume of the solid formed by revolving the region bounded by $y = \sqrt{x}$, the x-axis, and $x=4$ about the x-axis.

1. **Radii:** The radius of each disk is $R(x) = \sqrt{x}$.
2. **Set up integral:** Limits are from $x=0$ to $x=4$. $V = \int_0^4 \pi (\sqrt{x})^2 dx = \int_0^4 \pi x dx$
3. **Evaluate:** $V = \pi [x^2/2]_0^4 = \pi [(4^2)/2 - (0^2)/2] = \pi [16/2 - 0] = 8\pi$. The volume is 8π cubic units.

KEY TAKEAWAYS

- Integration by substitution simplifies integrals by reversing the chain rule, transforming them into simpler forms.
- Riemann sums approximate the area under a curve using rectangles, leading to the formal definition of the definite integral.
- The definite integral represents the net signed area between a function and the x-axis over a given interval.
- The Fundamental Theorem of Calculus provides an efficient method to evaluate definite integrals using antiderivatives.
- Areas between two curves are found by integrating the difference between the upper and lower functions.
- The disk and washer methods use definite integrals to calculate the volumes of solids formed by revolving a 2D region around an axis.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. Explain the purpose of integration by substitution.
2. How do Riemann sums relate to the definite integral?
3. State the Fundamental Theorem of Calculus Part 2 and explain its significance.
4. Describe the general procedure for finding the area between two curves.
5. When would you use the disk method versus the washer method for calculating volumes of revolution?

CHAPTER 5

Calculus Reference and Practice

This chapter serves as a vital consolidation point for your calculus path. It brings together the essential rules for derivatives and integrals, providing a concise reference that you will use repeatedly throughout your studies and beyond. Mastery of these fundamental rules is non-negotiable for success in calculus.

Beyond just knowing the rules, this chapter emphasizes the critical importance of active practice. Calculus is a skill developed through consistent problem-solving, not passive observation. We will outline effective strategies for studying lectures and drilling problems to build true fluency and confidence in applying calculus concepts.

Essential Calculus Rules

Covers: Derivative & Integral Cheat Sheet

To truly master calculus, you must commit its core rules to memory. These rules are the foundational tools you will use to analyze functions, rates of change, and accumulation. Understanding them deeply allows you to approach complex problems with confidence.

Derivatives

Derivatives measure the instantaneous rate of change of a function. Here are the most frequently used rules:

- **Power Rule:** This rule applies to functions of the form x^n . To differentiate, bring the exponent down as a coefficient and reduce the exponent by one.

$$\frac{d}{dx}[x^n] = n x^{(n-1)}$$

For example, the derivative of x^5 is $5x^4$. The derivative of x (which is x^1) is $1x^0$, or 1 .

- **Product Rule:** Used when differentiating a product of two functions, say $f(x)g(x)$. The rule states you take the derivative of the first function times the second, plus the first function times the derivative of the second.

$$(fg)' = f'g + fg'$$

Consider $x^2 \sin(x)$. Here $f=x^2$ and $g=\sin(x)$. $f'=2x$ and $g'=\cos(x)$. So, $(x^2 \sin(x))' = (2x)\sin(x) + x^2(\cos(x))$.

- **Quotient Rule:** Applied when differentiating a ratio of two functions, $f(x)/g(x)$. This rule is often remembered as 'low d-high minus high d-low over low squared'.

$$(f/g)' = (f'g - fg') / g^2$$

For $\sin(x)/x$, $f=\sin(x)$ and $g=x$. $f'=\cos(x)$ and $g'=1$. So, $(\sin(x)/x)' = (\cos(x) * x - \sin(x) * 1) / x^2$.

- **Chain Rule:** Essential for differentiating composite functions, $f(g(x))$. You differentiate the 'outer' function, keeping the 'inner' function intact, then multiply by the derivative of the 'inner' function.

$$d/dx[f(g(x))] = f'(g(x)) g'(x)$$

For $\sin(x^2)$, the outer function is $\sin(u)$ and the inner is $u=x^2$. The derivative is $\cos(x^2) * (2x)$.

- **Common Derivatives:** These functions appear frequently and their derivatives are straightforward.

$$\begin{aligned} d/dx[e^x] &= e^x \\ d/dx[\ln x] &= 1/x \\ d/dx[\sin x] &= \cos x \\ d/dx[\cos x] &= -\sin x \\ d/dx[\tan x] &= \sec^2 x \end{aligned}$$

Integrals

Integrals represent the accumulation of a quantity or the area under a curve. They are, in a fundamental sense, the inverse operation of differentiation.

- **Power Rule:** This rule reverses the derivative power rule. Add one to the exponent and divide by the new exponent. Remember to add the constant of integration, C .

$$\text{integral } x^n dx = x^{(n+1)}/(n+1) + C \text{ (for } n \neq -1)$$

For example, $\text{integral } x^3 dx = x^4/4 + C$.

- **Special Cases:**

$$\begin{aligned} \text{integral } 1/x dx &= \ln|x| + C \\ \text{integral } e^x dx &= e^x + C \\ \text{integral } \cos x dx &= \sin x + C \\ \text{integral } \sin x dx &= -\cos x + C \end{aligned}$$

- **u-substitution:** This technique is crucial for integrating composite functions and effectively reverses the Chain Rule. It involves identifying a part of the integrand as u and its derivative as du , transforming the integral into a simpler form. For example, in $\text{integral } 2x \cos(x^2) dx$, if we let $u = x^2$, then $du = 2x dx$. The integral becomes $\text{integral } \cos(u) du$, which is $\sin(u) + C$, or $\sin(x^2) + C$.

The Big Idea

At its core, the **derivative** provides the instantaneous rate of change or the slope of a tangent line. The **integral** calculates the total accumulation or the area under a curve. The **Fundamental Theorem of Calculus** elegantly connects these two concepts, stating that they are inverse operations. This profound relationship is what makes calculus such a powerful tool.

Strategies for Mastery Through Practice

Covers: Practice Until Fluent

Calculus is not a spectator sport. You cannot learn it effectively by simply watching someone else solve problems. True mastery comes from actively engaging with the material, getting your hands dirty with calculations, and working through challenges yourself. This section outlines a proven strategy for converting theoretical understanding into practical fluency.

The Active Learning Imperative

Many students find that concepts seem clear during a lecture, but then vanish when faced with a blank page. This gap between understanding and application is common. The only way to close it is through consistent, deliberate practice. Think of calculus as a skill, like playing a musical instrument or learning a sport. You improve by doing, by repeating movements, and by refining your technique.

How to Study a Lecture Effectively

Professor Leonard's lectures are designed to teach you the material thoroughly. To maximize your learning from each lecture, follow this three-step process:

- 1. Watch Professor Leonard Work an Example:** Pay close attention to the steps, reasoning, and any nuances Professor Leonard highlights. Understand *why* each step is taken, not just *what* the step is. Don't try to write notes on every detail during this initial watch; focus on comprehension.
- 2. Pause and Redo It Yourself on Paper:** Immediately after watching an example, pause the lecture. Without looking back at the solution, try to solve the exact same problem on your own paper. This step is crucial. It forces you to recall the method, apply the rules, and identify any points of confusion. If you get stuck, glance back at the solution only for the specific step you need, then try to continue on your own. The goal is to complete the problem independently.
- 3. Do 5 Similar Problems Before Moving On:** Once you successfully redo the lecture example, find at least five similar problems. These can come from your textbook, online exercise collections, or other reliable math resources. Work through these problems diligently. This repetition, often called 'reps,' solidifies your understanding, helps you recognize patterns, and builds speed and accuracy. It also exposes you to variations of the problem type, preparing you for different scenarios.

Why This Method Works

This structured approach to practice helps in several ways:

- **Immediate Application:** Redoing the example immediately reinforces the lesson. Your memory is fresh, and you can quickly identify where your understanding might be weak.
- **Skill Building:** Solving similar problems builds muscle memory for the procedures. It moves the knowledge from your short-term memory to long-term retention.
- **Problem Recognition:** With enough practice, you start to recognize problem types and instinctively know which rules or techniques to apply.
- **Confidence:** Each successfully solved problem builds your confidence, making you less intimidated by new challenges.
- **Identifying Weaknesses:** When you struggle with a problem, it clearly signals an area where you need more review or practice. This allows you to target your study efforts efficiently.

Do not skip the practice step. Concepts that feel obvious during a lecture can quickly become elusive on a blank page. Consistent, active problem-solving is the bridge to true calculus fluency.

KEY TAKEAWAYS

- Mastering calculus requires memorizing and understanding the fundamental rules for derivatives and integrals.
- Derivatives represent instantaneous rates of change, while integrals calculate accumulation or area under a curve.
- The Fundamental Theorem of Calculus establishes derivatives and integrals as inverse operations.
- Active problem-solving, not passive observation, is the most effective way to learn and internalize calculus concepts.
- A structured practice method involves watching an example, redoing it independently, and then solving multiple similar problems.
- Consistent repetition builds fluency, reinforces understanding, and helps bridge the gap between theoretical knowledge and practical application.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. What is the conceptual difference between a derivative and an integral?
2. State the Power Rule for derivatives and provide a simple example.
3. When is the Chain Rule typically applied, and how does it relate to u-substitution?
4. Describe the recommended three-step process for studying a calculus lecture effectively.
5. Why is active problem-solving considered more effective for learning calculus than just watching solutions?

CHAPTER 6

Final Assessment: Calculus 1

This chapter prepares you for the comprehensive final assessment in Calculus 1. This assessment is designed to evaluate your mastery of all core concepts covered throughout the course. It serves as a crucial checkpoint to confirm your understanding of limits, derivatives, and integrals.

Treat this assessment as a diagnostic tool. It will highlight your strengths and identify any areas that may require further review. A strong performance here demonstrates readiness to apply calculus principles to a wide range of problems.

What You Should Have Mastered

Covers: Final Assessment: Calculus 1 (quiz, 4 min)

By this point in the course, you should possess a solid understanding of all fundamental Calculus 1 concepts. This includes a firm grasp of limits and continuity, covering topics like evaluating limits algebraically and graphically, understanding infinite limits, and identifying points of discontinuity.

Mastery of derivatives is also essential. You should be proficient with all derivative rules, including the power rule, product rule, quotient rule, and chain rule. You must also understand how to apply derivatives to solve problems involving rates of change, related rates, and optimization. This requires a strong foundation in algebraic manipulation and trigonometric identities.

Finally, you should be able to apply integration techniques. This includes finding antiderivatives, using the Fundamental Theorem of Calculus, and applying definite integrals to calculate areas under curves and other quantities. Your ability to connect these concepts and solve multi-step problems is key. Remember that calculus builds upon itself; a weakness in one area can affect your performance in others.

Approaching the Final Assessment

Covers: Final Assessment: Calculus 1 (quiz, 4 min)

Approach the final assessment with a clear strategy. First, ensure you have reviewed all previous modules thoroughly. Revisit your notes, textbook examples, and practice problems from each chapter. Focus on understanding the *why* behind each method, not just memorizing formulas.

During the assessment itself, manage your time effectively. Read each question carefully to understand what is being asked. Do not rush through problems. If a question involves multiple steps, break it down and work through each part systematically. Even for multiple-choice questions, work out the solution completely before selecting an answer. This minimizes careless errors and reinforces your problem-solving process.

Common Mistakes and How to Avoid Them

Covers: Final Assessment: Calculus 1 (quiz, 4 min)

Students often make similar errors in calculus assessments. Many mistakes stem from basic algebra and trigonometry. Reviewing these prerequisite skills can prevent many unnecessary errors. For example, be careful with signs, fractions, and exponent rules.

Another common issue is misapplying derivative or integral rules. Double-check your work when using the chain rule, product rule, or quotient rule. Ensure you are using the correct integration technique for the given problem. Neglecting to consider domain restrictions for functions, especially when dealing with limits or derivatives, can also lead to incorrect answers. Always take your time and verify each step of your calculation. A moment of careful thought can save you from a significant error.

Interpreting Your Results

Covers: Final Assessment: Calculus 1 (quiz, 4 min)

Your score on the final assessment provides valuable feedback. A high score indicates a strong grasp of the course material. It confirms your readiness to move forward.

If your score is lower than expected, do not be discouraged. Instead, use the results as a guide for further study. Review the specific questions you answered incorrectly. Identify the underlying concepts that caused difficulty. Did you struggle with limits, derivative applications, or integration techniques? Go back to the corresponding lessons and examples in those modules. Work through additional practice problems. The goal is to solidify your understanding in those specific areas. This targeted review is far more effective than simply re-reading everything.

KEY TAKEAWAYS

- The final assessment evaluates your comprehensive understanding of Calculus 1 concepts.
- Thorough review of limits, derivatives, and integrals from all modules is essential for success.
- Effective time management and careful reading of questions are crucial during the assessment.
- Many common errors stem from algebraic mistakes or misapplication of calculus rules.
- Use your assessment results to identify specific areas needing further study and targeted review.
- A strong foundation in algebra and trigonometry directly supports success in calculus.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. What are the three main conceptual pillars covered in a standard Calculus 1 course?
2. Why is a strong understanding of algebra and trigonometry considered a prerequisite for Calculus 1?

3. How should you approach a comprehensive final assessment in calculus to maximize your chances of success?
4. What is the most effective way to use your results from the final assessment?
5. Name two common types of errors students make in calculus problems.

REFERENCE

Glossary

Antiderivative

A function $F(x)$ whose derivative is $f(x)$ (i.e., $F'(x) = f(x)$).

Chain rule

A rule for differentiating composite functions: $(f(g(x)))' = f'(g(x)) * g'(x)$.

Concavity

The direction of the curve's opening, either concave up (opens upward) or concave down (opens downward).

Constant of integration

The arbitrary constant C that appears in an indefinite integral, representing the family of antiderivatives.

Continuity

A property of a function where there are no abrupt changes, jumps, or holes in its graph.

Critical number

A point c in the domain of f where $f'(c) = 0$ or $f'(c)$ is undefined.

Definite integral

The limit of Riemann sums, representing the net signed area between a function's graph and the x -axis over an interval.

Derivative

The instantaneous rate of change of a function with respect to its variable, geometrically representing the slope of the tangent line.

Disk method

A technique for calculating the volume of a solid of revolution where the revolved region is flush against the axis of revolution.

Domain

The set of all possible input values for which a function is defined.

Function

A relation where each input has exactly one output.

Fundamental theorem of calculus

A theorem that links the concepts of differentiating a function and integrating a function, showing them to be inverse operations.

Fundamental theorem of calculus (ftc)

A theorem that connects differentiation and integration, providing a method to evaluate definite integrals using antiderivatives.

Horizontal asymptote

A horizontal line $y = L$ that the graph of a function approaches as x approaches positive or negative infinity.

Implicit differentiation

A technique for differentiating equations that implicitly define a function, typically by differentiating both sides with respect to x and solving for dy/dx .

Indefinite integral

The family of all antiderivatives of a function $f(x)$, denoted $\int f(x) dx = F(x) + C$.

Inflection point

A point on the graph where the concavity of the function changes.

Integral

A mathematical operation that calculates the total accumulation of a quantity or the area under a curve of a function.

Integration by substitution

A technique for evaluating integrals by changing the variable of integration to simplify the integrand.

Limit

The value that a function approaches as the input approaches some value.

One-sided limit

The value a function approaches as the input approaches a point from either the left or the right.

Optimization

The process of finding the maximum or minimum value of a function, often subject to constraints.

Power rule

A fundamental differentiation rule stating that the derivative of x^n is $n \cdot x^{(n-1)}$.

Product rule

A rule for differentiating the product of two functions: $(f \cdot g)' = f'g + fg'$.

Quotient rule

A rule for differentiating the quotient of two functions: $(f/g)' = (f'g - fg') / g^2$.

Range

The set of all possible output values of a function.

Related rates

Problems involving finding the rate of change of one quantity by relating it to other quantities whose rates of change are known, often solved using implicit differentiation with respect to time.

Riemann sum

An approximation of the area under a curve, formed by summing the areas of a finite number of rectangles.

Secant line

A line connecting two points on a curve.

Tangent line

A line that touches a curve at a single point and has the same slope as the curve at that point.

U-substitution

A technique for integration that simplifies complex integrals by changing the variable of integration, effectively reversing the Chain Rule.

Washer method

A technique for calculating the volume of a solid of revolution with a hole, used when there is a gap between the revolved region and the axis of revolution.

REFERENCE

Answer Key

Chapter 1: Section 1

1. What are the three conditions for a function $f(x)$ to be continuous at a point $x = c$?

For a function $f(x)$ to be continuous at $x = c$, three conditions must be met. First, $f(c)$ must be defined. Second, the limit of $f(x)$ as x approaches c must exist. Third, the limit of $f(x)$ as x approaches c must be equal to $f(c)$.

2. Explain the difference between the slope of a secant line and the slope of a tangent line.

A secant line connects two distinct points on a curve, and its slope represents the average rate of change between those two points. A tangent line touches a curve at a single point and represents the instantaneous rate of change at that specific point. The slope of a tangent line is found by taking the limit of the slopes of secant lines as the two points converge.

3. When does the limit of a function $f(x)$ as x approaches c not exist?

The limit of $f(x)$ as x approaches c does not exist if the left-hand limit and the right-hand limit are not equal. This can occur if the function approaches different values from each side, if it grows without bound (approaches infinity or negative infinity), or if it oscillates infinitely near c .

4. Given $f(x) = x^2 + 1$ and $g(x) = 2x - 3$, find $(f \circ g)(x)$ and state its domain.

To find $(f \circ g)(x)$, substitute $g(x)$ into $f(x)$. So, $f(g(x)) = f(2x - 3) = (2x - 3)^2 + 1$. Expanding this gives $4x^2 - 12x + 9 + 1 = 4x^2 - 12x + 10$. Since both $f(x)$ and $g(x)$ are polynomials, their composition is defined for all real numbers. Thus, the domain is $(-\infty, \infty)$.

5. What is the angle of inclination of a line with a slope of $m = 1$?

The angle of inclination θ is related to the slope m by the formula $\tan(\theta) = m$. If $m = 1$, then $\tan(\theta) = 1$. The angle θ in the interval $[0, \pi)$ (or $[0, 180)$ degrees) for which this is true is $\pi/4$ radians, or 45 degrees.

Chapter 2: The Derivative: Rules and Applications

1. What does the derivative of a function represent both geometrically and as a rate of change?

Geometrically, the derivative represents the slope of the tangent line to the function's graph at a specific point. As a rate of change, it measures the instantaneous rate at which the output of the function changes with respect to its input.

2. Explain when you would use the Product Rule versus the Chain Rule, and provide a simple example for each.

The Product Rule is used when you have two functions multiplied together, for example, $d/dx [x^2 \cdot \sin(x)]$. The Chain Rule is used for composite functions, where one function is 'inside' another, for example, $d/dx [\sin(x^2)]$.

3. List the derivatives of the six basic trigonometric functions.

The derivatives are: $d/dx [\sin(x)] = \cos(x)$, $d/dx [\cos(x)] = -\sin(x)$, $d/dx [\tan(x)] = \sec^2(x)$, $d/dx [\cot(x)] = -\csc^2(x)$, $d/dx [\sec(x)] = \sec(x)\tan(x)$, and $d/dx [\csc(x)] = -\csc(x)\cot(x)$.

4. Describe the general procedure for solving a related rates problem.

First, draw a diagram and label variables. Second, write an equation relating the quantities. Third, differentiate the equation implicitly with respect to time. Finally, substitute known values and solve for the desired unknown rate.

5. Find the derivative of $f(x) = (3x^2 - 5)^4$.

This requires the Chain Rule. Let the outer function be u^4 and the inner function be $3x^2 - 5$. The derivative of the outer is $4u^3$, and the derivative of the inner is $6x$. So, $f'(x) = 4(3x^2 - 5)^3 \cdot (6x) = 24x(3x^2 - 5)^3$.

Chapter 3: Analyzing Functions and Introduction to Integration

1. How do the first and second derivatives help in sketching the graph of a function?

The first derivative, $f'(x)$, tells us where the function is increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$). It also helps identify local maxima and minima using the First Derivative Test. The second derivative, $f''(x)$, indicates the concavity of the function: concave up ($f''(x) > 0$) or concave down ($f''(x) < 0$). It also helps locate inflection points where concavity changes and can be used in the Second Derivative Test for extrema.

2. Explain the relationship between Rolle's Theorem and the Mean Value Theorem.

Rolle's Theorem is a special case of the Mean Value Theorem. Both require a function to be continuous on a closed interval and differentiable on the open interval. The Mean Value Theorem states that there exists a point where the instantaneous rate of change equals the average rate of change over the interval. Rolle's Theorem adds the condition that the function values at the endpoints are equal, which implies the average rate of change is zero, so there must be a point where the instantaneous rate of change is also zero.

3. What is a horizontal asymptote, and how is it found using limits?

A horizontal asymptote is a horizontal line $y = L$ that the graph of a function approaches as the input x tends towards positive or negative infinity. It is found by evaluating the limits $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$. If either of these limits equals a finite value L , then $y = L$ is a horizontal asymptote.

4. Outline the general steps for solving an optimization problem.

First, identify the quantity to be optimized and define variables. Next, write an objective function for this quantity and identify any constraint equations. Use the constraints to express the objective function in terms of a single variable. Then, find the critical numbers of this function by setting its derivative to zero. Finally, test the critical numbers and endpoints of the domain to determine the absolute maximum or minimum value, and state the answer in context.

5. What is an antiderivative, and why is there a 'constant of integration' when finding indefinite integrals?

An antiderivative of a function $f(x)$ is a function $F(x)$ such that its derivative $F'(x)$ equals $f(x)$. There is a

constant of integration, C , because the derivative of any constant is zero. Therefore, if $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$ for any real constant C is also an antiderivative. The indefinite integral represents this entire family of functions.

Chapter 4: Integration and Its Applications

1. Explain the purpose of integration by substitution.

Integration by substitution is used to simplify integrals that are difficult to solve directly. It works by transforming the integral into a simpler form using a new variable, effectively reversing the chain rule from differentiation. This makes the integration process more manageable.

2. How do Riemann sums relate to the definite integral?

Riemann sums approximate the area under a curve by dividing it into rectangles and summing their areas. The definite integral is formally defined as the limit of these Riemann sums as the number of rectangles approaches infinity and their width approaches zero, providing the exact net signed area.

3. State the Fundamental Theorem of Calculus Part 2 and explain its significance.

The Fundamental Theorem of Calculus Part 2 states that if F is an antiderivative of f , then $\int_a^b f(x) dx = F(b) - F(a)$. Its significance is that it provides a direct and efficient way to evaluate definite integrals by finding an antiderivative and evaluating it at the limits, rather than using the complex limit of Riemann sums.

4. Describe the general procedure for finding the area between two curves.

To find the area between two curves, first sketch them to identify the upper and lower functions. Next, find their intersection points to determine the limits of integration. Finally, set up and evaluate the definite integral of the difference between the upper function and the lower function over the determined interval.

5. When would you use the disk method versus the washer method for calculating volumes of revolution?

You use the disk method when the region being revolved is flush against the axis of revolution, resulting in a solid without a hole. The washer method is used when there is a gap between the region and the axis of revolution, creating a solid with a hole in the center.

Chapter 5: Calculus Reference and Practice

1. What is the conceptual difference between a derivative and an integral?

A derivative measures the instantaneous rate of change or slope of a function at a specific point. An integral, conversely, calculates the total accumulation or the area under the curve of a function over an interval. They are inverse operations, as described by the Fundamental Theorem of Calculus.

2. State the Power Rule for derivatives and provide a simple example.

The Power Rule for derivatives states that $d/dx[x^n] = n x^{(n-1)}$. For example, if you want to find the derivative of x^7 , you would apply the rule to get $7x^{(7-1)}$, which simplifies to $7x^6$.

3. When is the Chain Rule typically applied, and how does it relate to u-substitution?

The Chain Rule is applied when differentiating composite functions, meaning a function within

another function, such as $\sin(x^2)$. It involves differentiating the outer function and multiplying by the derivative of the inner function. u-substitution in integration is the inverse process of the Chain Rule, used to simplify integrals of composite functions by transforming them into a simpler form.

4. Describe the recommended three-step process for studying a calculus lecture effectively.

The recommended process involves three steps: first, watch Professor Leonard work an example to understand the concept. Second, pause the lecture and immediately redo that exact example yourself on paper without looking at the solution. Third, solve at least five similar problems from other resources before moving on, to solidify your understanding and build fluency.

5. Why is active problem-solving considered more effective for learning calculus than just watching solutions?

Active problem-solving is more effective because it forces you to apply concepts, recall rules, and identify your own areas of confusion. Watching solutions is passive and doesn't build the necessary skills or intuition. By actively solving problems, you bridge the gap between theoretical understanding and practical application, leading to deeper learning and retention.

Chapter 6: Final Assessment: Calculus 1

1. What are the three main conceptual pillars covered in a standard Calculus 1 course?

The three main conceptual pillars are limits and continuity, derivatives and their applications, and integrals and their applications. These topics build upon each other to form the foundation of calculus.

2. Why is a strong understanding of algebra and trigonometry considered a prerequisite for Calculus 1?

Algebra and trigonometry are foundational because calculus operations often involve complex algebraic manipulations and trigonometric identities. Errors in these prerequisite areas can lead to incorrect calculus solutions, even if the calculus concepts are understood.

3. How should you approach a comprehensive final assessment in calculus to maximize your chances of success?

To maximize success, review all course material thoroughly, practice diverse problem types, and manage time effectively during the assessment. Read each question carefully and work through problems systematically, verifying each step.

4. What is the most effective way to use your results from the final assessment?

The most effective way is to use the results diagnostically. Identify specific topics or question types where you made errors. Then, return to those particular lessons or modules for targeted review and additional practice, rather than simply re-studying everything.

5. Name two common types of errors students make in calculus problems.

Two common types of errors include algebraic mistakes, such as sign errors or incorrect exponent manipulation, and misapplying calculus rules, such as using the wrong derivative or integral formula for a given function.

CLOSING

Where to Go Next

After completing Calculus 1, consolidate your skills by practicing additional problems. Work through exercises from textbooks, or explore online problem sets. Consider applying calculus concepts to problems in physics, engineering, or economics to see its practical utility.

The natural progression from Calculus 1 is often Calculus 2, which covers sequences, series, and advanced integration techniques. Other related subjects include differential equations, linear algebra, and multivariable calculus.

Continue to build your mathematical reasoning and problem-solving abilities. These skills are valuable across many disciplines and will serve you well in future academic and professional pursuits.



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