

OFFICIAL COURSE STUDY GUIDE

MIT 18.01: Single Variable Calculus

The full MIT OpenCourseWare calculus lectures.

SCIENCE & MATH · ADVANCED · 5 MODULES · 28 HR 32 MIN

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MIT 18.01: Single Variable Calculus: The Complete Study Guide

2026 Edition

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Contents

<i>About This Course</i>	4
CHAPTER 1 Introduction to Differentiation	6
CHAPTER 2 Applications of Differentiation	13
CHAPTER 3 Applications of Derivatives and Introduction to Antidifferentiation	21
CHAPTER 4 Techniques of Integration and Infinite Series	29
CHAPTER 5 Taylor and Maclaurin Series	36
<i>Glossary</i>	41
<i>Answer Key</i>	44
<i>Where to Go Next</i>	48

INTRODUCTION

About This Course

Calculus is the language of change. It provides essential tools for understanding motion, growth, decay, and optimization across science, engineering, and economics. Mastering its principles opens doors to advanced study and practical problem-solving.

This study guide supports "MIT 18.01: Single Variable Calculus," a comprehensive course based on the renowned MIT OpenCourseWare lectures. You will learn core concepts including differentiation, integration, limits, and series. This material is presented with university-level rigor.

MegaCourse has structured these lectures into a clear learning path of 5 modules and 35 lessons. This guide complements the video instruction, offering additional context, practice, and review opportunities. It helps you solidify your understanding of each topic.

By consistently engaging with the course videos and this guide, you will develop a robust analytical framework. This framework prepares you to approach complex mathematical challenges with confidence.

Course Facts

FORMAT	5 modules · 35 lessons
VIDEO INSTRUCTION	About 28 hr 32 min
LEVEL	Advanced
CATEGORY	Science & Math
INSTRUCTOR	MIT OpenCourseWare
CERTIFICATE	Included with course completion on MegaCourse
COURSE HOME	megacourse.app/courses/mit-18-01-single-variable-calculus

What You Will Learn

- Master differentiation and integration techniques.
- Apply calculus concepts to solve real-world problems.
- Understand the fundamental principles of limits and series.
- Analyze mathematical problems with university-level rigor.
- Develop a strong foundation for further study in mathematics and science.

Who This Course Is For

This course is designed for students who seek a thorough and rigorous introduction to single variable calculus. It serves those preparing for university-level mathematics, science, or engineering programs. The course is suitable for self-learners committed to mastering advanced mathematical concepts.

Before beginning, you should possess a strong foundation in both algebra and trigonometry. These subjects provide the essential framework upon which calculus is built. A solid grasp of these prerequisites will ensure you can fully engage with the course material.

Before You Start

- Possess a strong command of algebra.
- Demonstrate proficiency in trigonometry.
- Be comfortable with mathematical problem-solving.

How to Use This Guide

This study guide is designed to work hand-in-hand with the "MIT 18.01: Single Variable Calculus" video lectures. The course is organized into 5 modules and 35 lessons, totaling about 28 hours of instruction. We recommend a consistent study schedule to absorb the material effectively. Avoid rushing through complex topics.

For each lesson, first watch the corresponding video lecture. Pay close attention to the derivations and examples presented by the instructor. After watching, turn to this study guide. Review the key concepts, work through the provided examples, and complete the practice problems. This active engagement reinforces your learning.

The guide also includes module reviews and a glossary of terms. Use these sections to consolidate your understanding before moving to new modules. Regular review is crucial for retaining complex mathematical ideas over time.

CHAPTER 1

Introduction to Differentiation

This chapter introduces the fundamental concepts of differential calculus. You will learn about limits, the formal definition of the derivative, and essential rules for computing derivatives. We will explore how derivatives represent rates of change and slopes of tangent lines, providing powerful tools for analyzing functions. The chapter also covers practical applications such as related rates and linear approximations.

Mastering these foundational techniques is crucial for understanding advanced calculus topics and applying calculus to real-world problems. This material is presented with university rigor, preparing you for complex mathematical challenges.

Understanding Limits and the Derivative Definition

Covers: Lec 1 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 2 | MIT 18.01 Single Variable Calculus, Fall 2007

The foundation of calculus rests on the concept of a limit. A limit describes the value a function approaches as its input approaches some specific value. It is not necessarily the function's value *at* that point, but rather what it gets arbitrarily close to. For example, consider the function $f(x) = (x^2 - 1) / (x - 1)$. When $x = 1$, the function is undefined. However, as x gets closer and closer to 1 (from either side), $f(x)$ gets closer and closer to 2. We write this as $\lim_{x \rightarrow 1} [(x^2 - 1) / (x - 1)] = 2$.

The derivative builds directly upon the idea of a limit. It quantifies the instantaneous rate of change of a function. Imagine a curve on a graph. The average rate of change between two points is simply the slope of the secant line connecting them. If these two points are $(x, f(x))$ and $(x+h, f(x+h))$, the average rate of change is $[f(x+h) - f(x)] / h$. To find the instantaneous rate of change at a single point x , we let the distance h between the two points shrink to zero. This transforms the secant line into a tangent line at x . The slope of this tangent line is the derivative of the function at x .

The formal definition of the derivative of a function $f(x)$ is given by:

$$f'(x) = \lim_{h \rightarrow 0} [f(x+h) - f(x)] / h$$

This expression is known as the difference quotient. Common notations for the derivative include $f'(x)$, dy/dx , and $d/dx [f(x)]$.

Let us find the derivative of $f(x) = x^2$ using this definition.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} [(x+h)^2 - x^2] / h \\ &= \lim_{h \rightarrow 0} [x^2 + 2xh + h^2 - x^2] / h \\ &= \lim_{h \rightarrow 0} [2xh + h^2] / h \\ &= \lim_{h \rightarrow 0} [h(2x + h)] / h \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} [2x + h] \\
 &= 2x
 \end{aligned}$$

Thus, the derivative of x^2 is $2x$.

Fortunately, we do not always need to use the limit definition. Several rules simplify differentiation. The most fundamental is the **Power Rule**:

$$\frac{d}{dx} [x^n] = nx^{(n-1)}$$

This rule applies for any real number n . For example, $\frac{d}{dx} [x^3] = 3x^2$. Other basic rules include:

- **Constant Rule:** The derivative of a constant is zero. $\frac{d}{dx} [c] = 0$.
- **Constant Multiple Rule:** $\frac{d}{dx} [c \cdot f(x)] = c \cdot f'(x)$.
- **Sum and Difference Rule:** $\frac{d}{dx} [f(x) \pm g(x)] = f'(x) \pm g'(x)$.

Consider the function $g(x) = 5x^3 - 7x^2 + 4x - 10$. We can differentiate term by term:

$$\begin{aligned}
 g'(x) &= \frac{d}{dx} [5x^3] - \frac{d}{dx} [7x^2] + \frac{d}{dx} [4x] - \frac{d}{dx} [10] \\
 &= 5 \cdot (3x^2) - 7 \cdot (2x) + 4 \cdot (1) - 0 \\
 &= 15x^2 - 14x + 4
 \end{aligned}$$

A common mistake when starting out is confusing the average rate of change with the instantaneous rate of change. Remember, the derivative gives the rate at a single point, not over an interval. Always ensure you are applying the correct definition or rule for the specific problem. Practice differentiating various polynomial functions using these basic rules to build fluency.

Mastering Core Differentiation Rules

Covers: Lec 3 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 4 | MIT 18.01 Single Variable Calculus, Fall 2007

Beyond the basic power rule, several fundamental rules allow us to differentiate more complex functions. These are the product rule, quotient rule, and chain rule. Mastering these is crucial for nearly all differentiation tasks.

The **Product Rule** applies when you need to differentiate the product of two functions, $f(x)$ and $g(x)$. Its formula is:

$$\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

To illustrate, let $y = x^2 \cdot (3x + 1)$. Here, $f(x) = x^2$ and $g(x) = 3x + 1$. Then $f'(x) = 2x$ and $g'(x) = 3$. Applying the product rule:

$$\begin{aligned}
 \frac{dy}{dx} &= (2x)(3x + 1) + (x^2)(3) \\
 &= 6x^2 + 2x + 3x^2 \\
 &= 9x^2 + 2x
 \end{aligned}$$

The **Quotient Rule** is used for differentiating the ratio of two functions, $f(x) / g(x)$. The formula is:

$$d/dx [f(x)/g(x)] = [f'(x)g(x) - f(x)g'(x)] / [g(x)]^2$$

A common mnemonic is "low d high minus high d low, over low squared". Consider $y = (2x - 5) / (x^2 + 1)$. Here, $f(x) = 2x - 5$ and $g(x) = x^2 + 1$. Then $f'(x) = 2$ and $g'(x) = 2x$. Applying the quotient rule:

$$\begin{aligned} dy/dx &= [(2)(x^2 + 1) - (2x - 5)(2x)] / (x^2 + 1)^2 \\ &= [2x^2 + 2 - (4x^2 - 10x)] / (x^2 + 1)^2 \\ &= [2x^2 + 2 - 4x^2 + 10x] / (x^2 + 1)^2 \\ &= [-2x^2 + 10x + 2] / (x^2 + 1)^2 \end{aligned}$$

Be careful with the order of terms and the negative sign in the numerator.

The **Chain Rule** is arguably the most important differentiation rule. It applies when differentiating composite functions, that is, a function within a function, $f(g(x))$. The formula is:

$$d/dx [f(g(x))] = f'(g(x)) * g'(x)$$

This means you differentiate the "outside" function f , keeping the "inside" function $g(x)$ unchanged, and then multiply by the derivative of the "inside" function $g(x)$. For example, find dy/dx for $y = (x^3 + 2x)^5$. Here, the "outside" function is u^5 and the "inside" function is $u = x^3 + 2x$. The derivative of the outside function is $5u^4$. The derivative of the inside function is $3x^2 + 2$. Applying the chain rule:

$$dy/dx = 5(x^3 + 2x)^4 * (3x^2 + 2)$$

Failing to apply the chain rule is a very common error.

We can also find **higher order derivatives**. The second derivative, $f''(x)$ or d^2y/dx^2 , is the derivative of the first derivative. The third derivative, $f'''(x)$ or d^3y/dx^3 , is the derivative of the second derivative, and so on. These represent rates of change of rates of change. For instance, if $f(x)$ is position, $f'(x)$ is velocity, and $f''(x)$ is acceleration. To find $f''(x)$ for $f(x) = x^4 - 3x^2$: First derivative: $f'(x) = 4x^3 - 6x$. Second derivative: $f''(x) = d/dx [4x^3 - 6x] = 12x^2 - 6$.

Practice differentiating functions that combine these rules. For example, try $y = (x^2 + 1)^3 * (2x - 3)$. This requires both the product rule and the chain rule.

Applying Derivatives to Real Problems

Covers: Lec 5 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 6 | MIT 18.01 Single Variable Calculus, Fall 2007

Derivatives are powerful tools for solving problems in various fields. This section explores several key applications: implicit differentiation, related rates, linear approximation, and differentials.

Implicit Differentiation is used when a function y is not explicitly defined as a function of x , but rather is part of an equation relating x and y . For example, $x^2 + y^2 = 25$ describes a circle. To find

dy/dx , we differentiate both sides of the equation with respect to x , treating y as a function of x and applying the chain rule to any term involving y . Remember that $d/dx [y^n]$ becomes $n \cdot y^{n-1} \cdot dy/dx$.

Let us find dy/dx for $x^3 + y^3 = 6xy$. Differentiate each term with respect to x :

$$\begin{aligned} d/dx [x^3] + d/dx [y^3] &= d/dx [6xy] \\ 3x^2 + 3y^2 \cdot dy/dx &= 6 \cdot (1 \cdot y + x \cdot dy/dx) \quad (\text{Product rule on } 6xy) \\ 3x^2 + 3y^2 \cdot dy/dx &= 6y + 6x \cdot dy/dx \end{aligned}$$

Now, collect terms with dy/dx on one side and other terms on the other side:

$$\begin{aligned} 3y^2 \cdot dy/dx - 6x \cdot dy/dx &= 6y - 3x^2 \\ dy/dx (3y^2 - 6x) &= 6y - 3x^2 \\ dy/dx &= (6y - 3x^2) / (3y^2 - 6x) \\ dy/dx &= (2y - x^2) / (y^2 - 2x) \end{aligned}$$

A common mistake is forgetting to apply the chain rule to terms involving y .

Related Rates problems involve finding the rate at which one quantity is changing by relating it to other quantities whose rates of change are known. These problems typically involve implicit differentiation with respect to time, t . The general steps are:

1. Draw a diagram and label all quantities that are changing with variables.
2. Identify the given rates and the rate you need to find.
3. Write an equation that relates the variables.
4. Differentiate both sides of the equation with respect to t . Remember to use the chain rule for each variable.
5. Substitute all known values into the differentiated equation and solve for the unknown rate.

Consider a ladder 5 meters long leaning against a wall. The bottom of the ladder slides away from the wall at a rate of 0.5 m/s. How fast is the top of the ladder sliding down the wall when the bottom is 3 meters from the wall? Let x be the distance of the ladder's bottom from the wall, and y be the height of the ladder's top on the wall. The ladder length is 5. Equation: $x^2 + y^2 = 5^2 = 25$. Given: $dx/dt = 0.5$ m/s. When $x = 3$, we need dy/dt . First, find y when $x = 3$: $3^2 + y^2 = 25$, so $9 + y^2 = 25$, $y^2 = 16$, $y = 4$. Differentiate $x^2 + y^2 = 25$ with respect to t :

$$2x \cdot dx/dt + 2y \cdot dy/dt = 0$$

Substitute known values: $2(3)(0.5) + 2(4) \cdot dy/dt = 0$ $3 + 8 \cdot dy/dt = 0$ $8 \cdot dy/dt = -3$ $dy/dt = -3/8$ m/s. The negative sign indicates the top of the ladder is moving downwards.

Linear Approximation (or tangent line approximation) uses the tangent line to a function at a known point a to estimate the function's value at a nearby point x . The formula is:

$$L(x) = f(a) + f'(a)(x-a)$$

This is simply the equation of the tangent line at $x = a$. To approximate $\sqrt{4.1}$: Let $f(x) = \sqrt{x}$. Choose $a = 4$ because $\sqrt{4}$ is known. $f(a) = f(4) = \sqrt{4} = 2$. $f'(x) = 1 / (2\sqrt{x})$. $f'(a) = f'(4) = 1 / (2\sqrt{4}) = 1/4$. $x - a = 4.1 - 4 = 0.1$. $L(4.1) = 2 + (1/4)(0.1) = 2 + 0.025 = 2.025$. So, $\sqrt{4.1}$ is approximately 2.025.

Differentials are closely related to linear approximation. If $y = f(x)$, then the differential dy is defined as $dy = f'(x)dx$. Here, dx represents a small change in x , and dy represents the corresponding change in y along the tangent line. This provides an estimate for the actual change in y , denoted as Δy . Differentials are useful for estimating errors or changes in output given small changes in input.

Practice these applications by working through various problems. Pay close attention to setting up the equations correctly and applying the chain rule consistently.

Derivatives of Trigonometric Functions and Comprehensive Review

Covers: Lec 9 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 7: Exam 1 review | MIT 18.01 Single Variable Calculus, Fall 2007

This section introduces the derivatives of trigonometric functions and then provides a comprehensive review of all the differentiation techniques covered in this chapter. Understanding how to differentiate trigonometric functions is essential for many applications in physics and engineering.

The derivatives of the two fundamental trigonometric functions are:

$$\begin{aligned} d/dx [\sin(x)] &= \cos(x) \\ d/dx [\cos(x)] &= -\sin(x) \end{aligned}$$

Notice the negative sign for the derivative of $\cos(x)$. This is a common point of error.

We can derive the derivatives of other trigonometric functions using the quotient rule. For example, let us find the derivative of $\tan(x)$: Recall $\tan(x) = \sin(x) / \cos(x)$. Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Then $f'(x) = \cos(x)$ and $g'(x) = -\sin(x)$. Applying the quotient rule:

$$\begin{aligned} d/dx [\tan(x)] &= [\cos(x)\cos(x) - \sin(x)(-\sin(x))] / [\cos(x)]^2 \\ &= [\cos^2(x) + \sin^2(x)] / \cos^2(x) \\ &= 1 / \cos^2(x) \\ &= \sec^2(x) \end{aligned}$$

Similarly, the derivatives of the other trigonometric functions are:

$$\begin{aligned} d/dx [\cot(x)] &= -\csc^2(x) \\ d/dx [\sec(x)] &= \sec(x)\tan(x) \\ d/dx [\csc(x)] &= -\csc(x)\cot(x) \end{aligned}$$

It is important to memorize these derivatives.

When differentiating functions involving trigonometric terms, remember to combine these new rules with the previously learned product, quotient, and chain rules. For example, find dy/dx for $y = \sin(x^2 + 3x)$. This is a composite function, so we use the chain rule. The "outside" function is $\sin(u)$ and the

"inside" function is $u = x^2 + 3x$. $\frac{d}{du} [\sin(u)] = \cos(u)$. $\frac{d}{dx} [x^2 + 3x] = 2x + 3$. Applying the chain rule:

$$dy/dx = \cos(x^2 + 3x) * (2x + 3)$$

Another example, find dy/dx for $y = x^3 * \cos(x)$. This requires the product rule. Let $f(x) = x^3$ and $g(x) = \cos(x)$. $f'(x) = 3x^2$. $g'(x) = -\sin(x)$. Applying the product rule:

$$\begin{aligned} dy/dx &= (3x^2)\cos(x) + (x^3)(-\sin(x)) \\ &= 3x^2 \cos(x) - x^3 \sin(x) \end{aligned}$$

Comprehensive Review of Differentiation

This chapter has covered the foundational elements of differentiation. You began by understanding the concept of a limit and its role in defining the derivative as the instantaneous rate of change or the slope of a tangent line. We explored the formal limit definition of the derivative and then moved to more efficient methods using differentiation rules.

Key rules include:

- The **Power Rule** for x^n .
- The **Product Rule** for $f(x)g(x)$.
- The **Quotient Rule** for $f(x)/g(x)$.
- The **Chain Rule** for composite functions $f(g(x))$.

We also examined how to find higher order derivatives. The applications of these rules are extensive. You learned how to use **implicit differentiation** to find derivatives of equations not explicitly solved for y .

Related rates problems showed how to connect and solve for changing quantities over time. **Linear approximation** provided a method to estimate function values using tangent lines. Finally, **differentials** offered a way to quantify small changes. This section added the derivatives of all six **trigonometric functions**.

To solidify your understanding, attempt problems that combine multiple rules and concepts. For instance, find the equation of the tangent line to $y = \sin(x^2)$ at $x = \sqrt{\pi}$. This problem requires the chain rule, evaluating the function, and evaluating its derivative at a specific point. Consistent practice with varied problems is key to mastering differentiation.

KEY TAKEAWAYS

- The derivative represents the instantaneous rate of change of a function and the slope of the tangent line to its graph.
- The formal definition of the derivative involves a limit of the difference quotient.
- Mastering the power, product, quotient, and chain rules is essential for efficient differentiation.
- Implicit differentiation allows finding derivatives for equations where variables are not explicitly separated.
- Related rates problems use derivatives to connect and solve for unknown rates of change in dynamic scenarios.
- Linear approximation and differentials provide methods to estimate function values and changes using tangent lines.
- The derivatives of trigonometric functions like $\sin(x)$ and $\cos(x)$ are fundamental and combine with other rules.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. Explain the relationship between the slope of a secant line and the slope of a tangent line.
2. State the product rule and quotient rule for differentiation.
3. When is the chain rule necessary, and how is it applied?
4. Describe a scenario where implicit differentiation would be required.
5. What are the derivatives of $\sin(x)$ and $\cos(x)$?

CHAPTER 2

Applications of Differentiation

This chapter moves beyond the mechanics of finding derivatives and explores their powerful applications. You will learn how the rate of change of one quantity relates to the rate of change of another. We will use calculus to solve real-world problems involving maximizing or minimizing quantities, a technique known as optimization.

Also, this chapter introduces methods for approximating function values and finding roots of equations. You will also develop a comprehensive strategy for sketching the graph of a function by analyzing its first and second derivatives. These skills are fundamental for understanding and solving complex problems in various scientific and engineering fields.

Related Rates

Covers: Lec 10 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 11 | MIT 18.01 Single Variable Calculus, Fall 2007

Related rates problems involve finding the rate at which one quantity changes given the rates at which other related quantities change. The core idea is to identify a relationship between the variables, differentiate that relationship with respect to time, and then substitute known values.

The Strategy for Solving Related Rates Problems

1. **Read and Visualize:** Carefully read the problem. Draw a diagram if possible, labeling all quantities that change with time using variables. Label constant quantities with their numerical values.
2. **Identify Knowns and Unknowns:** List all given rates of change and quantities at a specific moment. Identify the rate you need to find.
3. **Formulate an Equation:** Write an equation that relates the variables from your diagram. This equation should hold true for all times, not just at a specific instant.
4. **Differentiate Implicitly:** Differentiate both sides of your equation with respect to time (t). Remember to use the chain rule for any variable that depends on t .
5. **Substitute and Solve:** Substitute all known values (variables and their rates of change) into the differentiated equation. Then, solve for the unknown rate.

Worked Example: The Expanding Circle

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outermost ripple is increasing at a rate of 1 foot per second. How fast is the total area A of the disturbed water increasing when the radius is 4 feet?

1. **Diagram:** Imagine a circle. Label its radius r and area A .
2. **Knowns/Unknowns:**
 - Known rate: $dr/dt = 1 \text{ ft/s}$

- Known quantity at an instant: $r = 4$ ft
 - Unknown rate: dA/dt
3. **Equation:** The area of a circle is $A = \pi r^2$.
4. **Differentiate:** Differentiate $A = \pi r^2$ with respect to time t :

$$\begin{aligned} dA/dt &= d/dt (\pi r^2) \\ dA/dt &= 2\pi r (dr/dt) \text{ (using the chain rule)} \end{aligned}$$

5. **Substitute and Solve:** Substitute the known values into the differentiated equation:

$$\begin{aligned} dA/dt &= 2\pi(4 \text{ ft})(1 \text{ ft/s}) \\ dA/dt &= 8\pi \text{ ft}^2/\text{s} \end{aligned}$$

The area is increasing at 8π square feet per second when the radius is 4 feet.

Common Mistakes

- **Substituting too early:** Do not substitute numerical values for variables that are changing *before* you differentiate. If you do, their derivatives will incorrectly become zero. Only substitute constants or values at the specific instant *after* differentiation.
- **Forgetting the Chain Rule:** Every variable that is a function of time t must have its derivative with respect to t included via the chain rule (e.g., $d(r^2)/dt = 2r dr/dt$).
- **Units:** Always include units in your final answer. They help verify the correctness of your calculation.

Optimization: Finding Maximum and Minimum Values

Covers: Lec 12 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 13 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 14 | MIT 18.01 Single Variable Calculus, Fall 2007

Optimization problems involve finding the absolute maximum or minimum value of a function. These problems are common in economics, engineering, and science, where we often seek to maximize profit, minimize cost, or find the most efficient design. The process relies on the fact that local extrema occur at critical points.

Optimization Procedure

1. **Understand the Problem:** Read carefully. Identify the quantity to be maximized or minimized. Draw a diagram if helpful.
2. **Formulate the Objective Function:** Define a function, let's call it $f(x)$, that represents the quantity you want to optimize. This function should depend on a single variable x .
3. **Determine the Domain:** Identify the feasible range of values for x . This might be an open interval (a, b) or a closed interval $[a, b]$.
4. **Find Critical Points:** Calculate the first derivative, $f'(x)$. Set $f'(x) = 0$ and solve for x . Also, find any x values where $f'(x)$ is undefined. These are your critical points.

5. Test Critical Points and Endpoints:

- **For a closed interval $[a, b]$:** Evaluate $f(x)$ at each critical point within the interval and at the endpoints a and b . The largest of these values is the absolute maximum, and the smallest is the absolute minimum.
- **For an open interval (a, b) or an unbounded domain:** Use the First Derivative Test or the Second Derivative Test to classify each critical point as a local maximum, local minimum, or neither. If there is only one local extremum in the domain, it is often the absolute extremum.

6. **State the Answer:** Clearly answer the question asked in the problem, including units if appropriate.

Worked Example: Maximizing Area

A farmer has 200 feet of fencing to enclose a rectangular area adjacent to a long barn. No fencing is needed along the barn wall. What dimensions will maximize the area of the enclosure?

1. **Understand:** Maximize the area of a rectangle. One side is the barn.
2. **Objective Function:** Let x be the length of the sides perpendicular to the barn, and y be the length of the side parallel to the barn. Area $A = xy$. We need A in terms of one variable. The total fencing is $2x + y = 200$. So, $y = 200 - 2x$. Substitute y into the area formula: $A(x) = x(200 - 2x) = 200x - 2x^2$.
3. **Domain:** Since x and y must be positive lengths, $x > 0$. Also, $200 - 2x > 0$, which means $200 > 2x$, or $x < 100$. So the domain is $(0, 100)$.
4. **Critical Points:** Find $A'(x)$:

$$A'(x) = 200 - 4x$$

Set $A'(x) = 0$:

$$\begin{aligned} 200 - 4x &= 0 \\ 4x &= 200 \\ x &= 50 \end{aligned}$$

This is the only critical point within the domain $(0, 100)$. 5. **Test:** Since the domain is an open interval, we use the First or Second Derivative Test. Let's use the Second Derivative Test:

$$A''(x) = -4$$

Since $A''(50) = -4 < 0$, the critical point $x = 50$ corresponds to a local maximum. Because it's the only critical point in the domain, it must be the absolute maximum. 6. **Answer:** When $x = 50$ feet, $y = 200 - 2(50) = 100$ feet. The dimensions that maximize the area are 50 feet by 100 feet. The maximum area is $50 * 100 = 5000$ square feet.

Common Mistakes

- **Not using a single variable:** Ensure your objective function depends on only one variable before differentiating. Use constraints to eliminate extra variables.

- **Ignoring the domain:** The domain of the function is crucial. It dictates whether endpoints need to be checked and helps determine if a local extremum is also an absolute extremum.
- **Not answering the question:** After finding the optimal x value, go back and ensure you've answered what the problem asked for (e.g., dimensions, maximum value, etc.).

Approximation Techniques: Linearization and Newton's Method

Covers: Lec 15 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 16 | MIT 18.01 Single Variable Calculus, Fall 2007

Calculus provides powerful tools for approximation. We can use derivatives to estimate function values or to find roots of complex equations.

Linear Approximation (Linearization)

Linear approximation uses the tangent line to a function's graph at a known point to estimate function values near that point. The tangent line is the best linear approximation of the function at that point.

- **Formula:** The equation of the tangent line to $f(x)$ at $x = a$ is:

$$L(x) = f(a) + f'(a)(x - a)$$

Here, $L(x)$ is the linear approximation of $f(x)$ near $x = a$.

- **Worked Example: Approximating Square Roots** Use linear approximation to estimate the value of $\sqrt{4.1}$.

1. **Identify $f(x)$ and a :** Let $f(x) = \sqrt{x}$. We know $f(4) = 2$, so choose $a = 4$. We want to approximate $f(4.1)$.

2. **Find $f'(x)$:** $f'(x) = d/dx (x^{1/2}) = (1/2)x^{-1/2} = 1/(2\sqrt{x})$.

3. **Evaluate $f(a)$ and $f'(a)$:**

- $f(4) = \sqrt{4} = 2$
- $f'(4) = 1/(2\sqrt{4}) = 1/(2 \cdot 2) = 1/4 = 0.25$

4. **Apply the Linearization Formula:**

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ L(x) &= 2 + 0.25(x - 4) \end{aligned}$$

5. **Estimate $f(4.1)$:** Substitute $x = 4.1$ into $L(x)$:

$$\begin{aligned} L(4.1) &= 2 + 0.25(4.1 - 4) \\ L(4.1) &= 2 + 0.25(0.1) \\ L(4.1) &= 2 + 0.025 \\ L(4.1) &= 2.025 \end{aligned}$$

So, $\sqrt{4.1}$ is approximately 2.025. (A calculator gives approximately 2.0248, showing the approximation is quite good.)

Newton's Method

Newton's Method is an iterative process used to find successively better approximations to the roots (or zeros) of a real-valued function. It starts with an initial guess and refines it using the tangent line.

- **Formula:** If x_n is an approximation for a root, the next approximation x_{n+1} is given by:

$$x_{n+1} = x_n - f(x_n) / f'(x_n)$$

The process is repeated until the approximations converge to a stable value.

- **Worked Example: Finding a Root** Use Newton's Method to find a root of $f(x) = x^2 - 2$, starting with an initial guess $x_0 = 1.5$.

1. Identify $f(x)$ and $f'(x)$:

- $f(x) = x^2 - 2$
- $f'(x) = 2x$

2. Apply the Formula Iteratively:

- **$n = 0$:**

$$\begin{aligned} x_1 &= x_0 - f(x_0) / f'(x_0) \\ x_1 &= 1.5 - ((1.5)^2 - 2) / (2 * 1.5) \\ x_1 &= 1.5 - (2.25 - 2) / 3 \\ x_1 &= 1.5 - 0.25 / 3 \\ x_1 &= 1.5 - 0.08333... \\ x_1 &\approx 1.416667 \end{aligned}$$

- **$n = 1$:**

$$\begin{aligned} x_2 &= x_1 - f(x_1) / f'(x_1) \\ x_2 &= 1.416667 - ((1.416667)^2 - 2) / (2 * 1.416667) \\ x_2 &= 1.416667 - (2.006945 - 2) / 2.833334 \\ x_2 &= 1.416667 - 0.006945 / 2.833334 \\ x_2 &\approx 1.416667 - 0.002451 \\ x_2 &\approx 1.414216 \end{aligned}$$

The root is $\sqrt{2} \approx 1.41421356$. Newton's Method quickly converges to the actual root.

Common Mistakes

- **Linearization:** Choosing an 'a' value that is too far from the point you want to approximate. The approximation is best close to 'a'.
- **Newton's Method:** A poor initial guess can lead to divergence or convergence to a different root. Also, if $f'(x_n) = 0$, the method fails.

Curve Sketching

Covers: Lec 18 | MIT 18.01 Single Variable Calculus, Fall 2007

Curve sketching is the process of using information from a function and its derivatives to accurately draw its graph. This involves systematically analyzing various features of the function.

Comprehensive Curve Sketching Strategy

1. **Domain:** Determine the domain of $f(x)$. This identifies where the function is defined.
2. **Intercepts:**
 - **y-intercept:** Set $x = 0$ and calculate $f(0)$.
 - **x-intercepts:** Set $f(x) = 0$ and solve for x . These can be difficult to find and might be skipped if too complex.
3. **Symmetry:** Check for even ($f(-x) = f(x)$) or odd ($f(-x) = -f(x)$) symmetry. This can reduce the amount of work.
4. **Asymptotes:**
 - **Vertical Asymptotes:** Occur where the denominator is zero and the numerator is non-zero for rational functions (e.g., $x = c$ if $\lim_{x \rightarrow c} f(x) = \pm\infty$).
 - **Horizontal Asymptotes:** Determined by $\lim_{x \rightarrow \pm\infty} f(x) = L$. If the limit exists, $y = L$ is a horizontal asymptote.
5. **First Derivative Analysis ($f'(x)$):**
 - **Critical Points:** Find $f'(x)$, set it to zero, and find where it is undefined. These points are potential local maxima, minima, or points of inflection.
 - **Intervals of Increase/Decrease:** Use a sign chart for $f'(x)$. Where $f'(x) > 0$, f is increasing. Where $f'(x) < 0$, f is decreasing.
 - **Local Extrema:** Apply the First Derivative Test at critical points to classify them as local maxima or minima.
6. **Second Derivative Analysis ($f''(x)$):**
 - **Possible Inflection Points:** Find $f''(x)$, set it to zero, and find where it is undefined. These are potential inflection points.
 - **Intervals of Concavity:** Use a sign chart for $f''(x)$. Where $f''(x) > 0$, f is concave up. Where $f''(x) < 0$, f is concave down.
 - **Inflection Points:** Points where concavity changes.
7. **Sketch the Graph:** Combine all the information gathered. Plot intercepts, asymptotes, local extrema, and inflection points. Then, connect the points, respecting the intervals of increasing/decreasing and concavity.

Worked Example: Sketching a Polynomial

Sketch the graph of $f(x) = x^3 - 3x^2 + 2$.

1. **Domain:** All real numbers $(-\infty, \infty)$.

2. **Intercepts:**

- y-intercept: $f(0) = 2$. So $(0, 2)$.
- x-intercepts: $x^3 - 3x^2 + 2 = 0$. This is hard to solve by hand. We will rely on derivatives.

3. **Symmetry:** No obvious even or odd symmetry.

4. **Asymptotes:** No vertical or horizontal asymptotes for a polynomial.

5. **First Derivative:**

- $f'(x) = 3x^2 - 6x = 3x(x - 2)$.
- Critical points: $f'(x) = 0$ when $3x = 0$ ($x = 0$) or $x - 2 = 0$ ($x = 2$).
- Sign chart for $f'(x)$:
- $x < 0$: $f'(-1) = 3(-1)(-3) = 9 > 0$ (Increasing)
- $0 < x < 2$: $f'(1) = 3(1)(-1) = -3 < 0$ (Decreasing)
- $x > 2$: $f'(3) = 3(3)(1) = 9 > 0$ (Increasing)
- Local extrema:
- At $x = 0$: $f(0) = 2$. Local maximum (from increasing to decreasing).
- At $x = 2$: $f(2) = 2^3 - 3(2)^2 + 2 = 8 - 12 + 2 = -2$. Local minimum (from decreasing to increasing).

6. **Second Derivative:**

- $f''(x) = 6x - 6 = 6(x - 1)$.
- Possible inflection point: $f''(x) = 0$ when $x - 1 = 0$ ($x = 1$).
- Sign chart for $f''(x)$:
- $x < 1$: $f''(0) = -6 < 0$ (Concave down)
- $x > 1$: $f''(2) = 6 > 0$ (Concave up)
- Inflection point: At $x = 1$, $f(1) = 1^3 - 3(1)^2 + 2 = 1 - 3 + 2 = 0$. So $(1, 0)$ is an inflection point.

7. **Sketch:** Plot $(0, 2)$ (local max), $(2, -2)$ (local min), $(1, 0)$ (inflection point, also an x-intercept), and the y-intercept $(0, 2)$. Connect these points, following the increasing/decreasing and concavity information. The graph increases to $(0, 2)$, decreases to $(2, -2)$, then increases again. It changes from concave down to concave up at $(1, 0)$.

Common Mistakes

- **Calculation errors:** Derivatives, especially the second derivative, can become complex. Double-check your algebra.

- **Confusing first and second derivative information:** Remember f' tells you about increasing/decreasing and local extrema; f'' tells you about concavity and inflection points.
- **Incomplete analysis:** Skipping steps like checking asymptotes or domain can lead to an inaccurate sketch.

KEY TAKEAWAYS

- Related rates problems connect the rates of change of multiple variables through a single equation, differentiated implicitly with respect to time.
- Optimization involves finding the absolute maximum or minimum of a function by analyzing its critical points and considering its domain.
- Linear approximation uses the tangent line to estimate function values near a known point, providing a simple yet effective estimation tool.
- Newton's Method is an iterative process that refines an initial guess to find the roots of a function by repeatedly using the tangent line.
- Curve sketching systematically combines information from a function's domain, intercepts, asymptotes, and its first and second derivatives to accurately draw its graph.
- The first derivative indicates where a function is increasing or decreasing and helps identify local extrema.
- The second derivative reveals the concavity of a function and helps locate inflection points.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. What is the key step in solving a related rates problem, and why is it important not to substitute numerical values too early?
2. Describe the primary role of the first derivative and the second derivative in curve sketching.
3. When solving an optimization problem on a closed interval, what points must you evaluate to find the absolute maximum and minimum?
4. Explain the concept behind linear approximation and provide its formula.
5. What is Newton's Method used for, and what information does it require to proceed iteratively?

CHAPTER 3

Applications of Derivatives and Introduction to Antidifferentiation

This chapter expands your understanding of derivatives beyond basic calculation. You will learn to apply differentiation to solve a wide array of problems, from finding optimal solutions in real-world scenarios to accurately sketching complex functions. These applications solidify your grasp of how derivatives describe rates of change and local behavior.

The chapter also introduces the fundamental concept of antidifferentiation. This marks a pivotal shift in your calculus studies, moving from finding the rate of change of a function to finding the function itself given its rate of change. You will begin to explore the inverse process of differentiation, setting the stage for integral calculus.

Applications of Derivatives: Rates, Approximations, and Root Finding

Covers: Lec 19 · Lec 20 · Lec 21

This section focuses on using derivatives to solve practical problems. We begin with related rates, a powerful method for analyzing how interconnected quantities change over time. Then, we explore how derivatives provide excellent approximations for functions, leading to methods for estimating values and finding roots.

Related Rates

Related rates problems involve finding the rate of change of one quantity given the rate of change of another quantity. The key is to recognize that all quantities are functions of time. You use implicit differentiation with respect to time to solve these problems.

Here is a general procedure:

1. **Read and Understand:** Identify what is known and what needs to be found. Draw a diagram if possible.
2. **Assign Variables:** Label all quantities involved.
3. **Formulate Equation:** Write an equation that relates the variables. This equation often comes from geometry or physics.
4. **Differentiate Implicitly:** Differentiate both sides of the equation with respect to time (t). Remember the Chain Rule.
5. **Substitute and Solve:** Plug in the known values for the variables and their rates of change. Solve for the unknown rate.

Example: A ladder 10 feet long leans against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom is 6 feet

from the wall?

Let x be the distance of the ladder's bottom from the wall, and y be the distance of the ladder's top from the ground. The ladder is 10 feet long. By the Pythagorean theorem: $x^2 + y^2 = 10^2$. We are given $dx/dt = 1$ ft/s. We need to find dy/dt when $x = 6$ ft.

First, find y when $x = 6$:

$$\begin{aligned}6^2 + y^2 &= 100 \\36 + y^2 &= 100 \\y^2 &= 64 \\y &= 8 \text{ ft}\end{aligned}$$

Now, differentiate $x^2 + y^2 = 100$ with respect to t : $2x (dx/dt) + 2y (dy/dt) = 0$

Substitute the known values: $x = 6$, $y = 8$, $dx/dt = 1$:

$$\begin{aligned}2(6)(1) + 2(8)(dy/dt) &= 0 \\12 + 16 (dy/dt) &= 0 \\16 (dy/dt) &= -12 \\dy/dt &= -12/16 = -3/4 \text{ ft/s}\end{aligned}$$

The negative sign indicates the top of the ladder is sliding *down*.

Linear and Quadratic Approximations

Derivatives provide a way to approximate function values near a known point. The **linear approximation** (or tangent line approximation) of a function $f(x)$ near $x = a$ is given by: $L(x) = f(a) + f'(a)(x - a)$. This approximation is essentially the equation of the tangent line to $f(x)$ at $x = a$. It is very useful for estimating values or understanding local behavior. The differential $dy = f'(x) dx$ is closely related, representing the change in y along the tangent line for a small change dx in x .

The **quadratic approximation** provides a more accurate estimate by including the second derivative: $Q(x) = f(a) + f'(a)(x - a) + (f''(a)/2)(x - a)^2$. This approximation curves with the function, offering a better fit over a larger interval than the linear approximation.

Example: Approximate $\sqrt{4.1}$ using linear approximation. Let $f(x) = \sqrt{x}$. We know $f(4) = 2$. So, $a = 4$. $f'(x) = 1/(2\sqrt{x})$. $f'(4) = 1/(2\sqrt{4}) = 1/4$. $L(x) = f(4) + f'(4)(x - 4)$. $L(x) = 2 + (1/4)(x - 4)$. For $x = 4.1$: $L(4.1) = 2 + (1/4)(4.1 - 4) = 2 + (1/4)(0.1) = 2 + 0.025 = 2.025$. The actual value of $\sqrt{4.1}$ is approximately 2.02484. The approximation is very close.

Newton's Method

Newton's Method is an iterative technique for finding approximations to the roots (zeros) of a real-valued function. It starts with an initial guess x_0 and refines it using the formula: $x_{n+1} = x_n - f(x_n)/f'(x_n)$. This formula is derived from the tangent line at x_n . The next approximation x_{n+1} is the x -intercept of this tangent line. The process repeats until successive approximations are sufficiently close.

*Consider finding a root of $f(x) = x^2 - 2$ (i.e., $\sqrt{2}$). Let $x_0 = 1.5$. $f'(x) = 2x$.

$$\begin{aligned}x_1 &= x_0 - f(x_0)/f'(x_0) = 1.5 - (1.5^2 - 2)/(2 \cdot 1.5) \\x_1 &= 1.5 - (2.25 - 2)/3 = 1.5 - 0.25/3 \\x_1 &= 1.5 - 0.0833\dots = 1.4167 \text{ (approx.)}\end{aligned}$$

The actual $\sqrt{2}$ is approximately 1.41421. Newton's method converges quickly. Be aware that Newton's Method can fail or converge slowly if the initial guess is poor, or if $f'(x_n)$ is close to zero.

Analyzing Function Behavior: Mean Value Theorem and Curve Sketching

Covers: Lec 22 · Lec 23

This section explores how derivatives reveal critical information about a function's shape and behavior. We will first discuss a fundamental theorem relating average and instantaneous rates of change, then apply derivative tests to sketch accurate graphs.

The Mean Value Theorem

The **Mean Value Theorem (MVT)** is a cornerstone of calculus. It states that for a function $f(x)$ that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , there exists at least one point c in (a, b) such that: $f'(c) = (f(b) - f(a))/(b - a)$. Geometrically, this means there is at least one point c where the tangent line to the curve $f(x)$ is parallel to the secant line connecting the points $(a, f(a))$ and $(b, f(b))$.

Rolle's Theorem is a special case of the MVT. If, in addition to the MVT conditions, $f(a) = f(b)$, then there exists at least one c in (a, b) such that $f'(c) = 0$. This means there is a horizontal tangent line somewhere between a and b .

The MVT is crucial for proving other theorems and understanding function properties. For example, if $f'(x) > 0$ on an interval, the MVT implies $f(x)$ is increasing on that interval.

Curve Sketching with Derivatives

Derivatives provide powerful tools for analyzing the shape of a graph. Combining information from the first and second derivatives allows for accurate curve sketching.

Here is a systematic approach to sketching a function $y = f(x)$:

1. Domain and Intercepts:

- Determine the domain of $f(x)$.
- Find x -intercepts by setting $f(x) = 0$.
- Find the y -intercept by calculating $f(0)$.

2. Asymptotes:

- **Vertical Asymptotes:** Occur where $f(x)$ approaches infinity (or negative infinity) as x approaches a finite value (typically where the denominator is zero for rational functions).
- **Horizontal Asymptotes:** Determine $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.

3. First Derivative Analysis:

- Calculate $f'(x)$.
- Find **critical points** by setting $f'(x) = 0$ or finding where $f'(x)$ is undefined.
- Use $f'(x)$ to determine intervals where $f(x)$ is **increasing** ($f'(x) > 0$) or **decreasing** ($f'(x) < 0$).
- Apply the **First Derivative Test** to classify local maxima and minima at critical points. A local maximum occurs if $f'(x)$ changes from positive to negative. A local minimum occurs if $f'(x)$ changes from negative to positive.

4. Second Derivative Analysis:

- Calculate $f''(x)$.
- Find **possible inflection points** by setting $f''(x) = 0$ or finding where $f''(x)$ is undefined.
- Use $f''(x)$ to determine intervals where $f(x)$ is **concave up** ($f''(x) > 0$) or **concave down** ($f''(x) < 0$).
- An **inflection point** occurs where the concavity changes.
- Apply the **Second Derivative Test** to classify local extrema: if $f'(c) = 0$ and $f''(c) > 0$, there is a local minimum at c . If $f'(c) = 0$ and $f''(c) < 0$, there is a local maximum at c . If $f''(c) = 0$, the test is inconclusive.

5. **Sketch the Graph:** Combine all this information to draw the curve. Plot intercepts, critical points, and inflection points. Show asymptotes and indicate increasing/decreasing and concavity.

Common Mistake: Confusing critical points (where $f'(x) = 0$ or undefined) with inflection points (where $f''(x) = 0$ or undefined). These concepts describe different aspects of a function's behavior. Critical points relate to local extrema, while inflection points relate to changes in concavity.

Advanced Applications: Optimization and L'Hopital's Rule

Covers: Lec 24 · Lec 25

This section works through two significant applications of derivatives. First, we tackle optimization problems, which involve finding the absolute maximum or minimum values of a function to solve real-world challenges. Second, we introduce L'Hopital's Rule, a powerful technique for evaluating limits of indeterminate forms.

Optimization Problems

Optimization problems involve finding the largest or smallest value of a quantity. This quantity is typically represented by a function, and the goal is to find its absolute maximum or minimum over a specific domain.

Here is a systematic approach to solving optimization problems:

1. **Understand the Problem:** Read carefully. Identify the quantity to be optimized (objective function) and any constraints.

2. **Draw a Diagram:** If applicable, sketch the situation and label relevant quantities with variables.
3. **Formulate the Objective Function:** Express the quantity to be optimized as a function of one or more variables.
4. **Identify Constraints and Reduce Variables:** Use any given constraints to express the objective function in terms of a single independent variable.
5. **Determine the Domain:** Find the feasible range of values for the independent variable. This often leads to a closed interval.
6. **Find Absolute Extrema:**
 - Calculate the first derivative of the objective function.
 - Find the critical points by setting the derivative to zero or finding where it is undefined within the domain.
 - Evaluate the objective function at the critical points and at the endpoints of the domain.
 - The largest of these values is the absolute maximum, and the smallest is the absolute minimum.
 - Alternatively, for an open interval, use the First or Second Derivative Test to classify local extrema, then consider the function's behavior as x approaches the interval boundaries.

Example: A farmer has 200 feet of fencing to enclose a rectangular area and divide it into two pens with a fence parallel to one side. What is the largest possible area?

Let the length of the field be L and the width be W . The total fencing used is $2L + 3W = 200$. This is our constraint. The area to maximize is $A = L * W$. This is our objective function.

From the constraint, $2L = 200 - 3W$, so $L = 100 - (3/2)W$. Substitute L into the area formula: $A(W) = (100 - (3/2)W) * W = 100W - (3/2)W^2$.

The domain for W is $0 < W < (200/3)$ (since L must be positive). Now, find the derivative of $A(W)$: $A'(W) = 100 - 3W$. Set $A'(W) = 0$ to find critical points:

$$\begin{aligned}100 - 3W &= 0 \\3W &= 100 \\W &= 100/3 \text{ feet}\end{aligned}$$

This critical point is within our domain. To confirm it is a maximum, use the Second Derivative Test: $A''(W) = -3$. Since $A''(W)$ is always negative, the function is concave down, meaning $W = 100/3$ corresponds to a local maximum, which is also the absolute maximum on this domain.

Now find L : $L = 100 - (3/2)(100/3) = 100 - 50 = 50$ feet. The maximum area is $A = L * W = 50 * (100/3) = 5000/3$ square feet.

Common Pitfall: Forgetting to check the endpoints of the domain if the domain is a closed interval. The absolute extremum can occur at an endpoint.

L'Hopital's Rule

L'Hopital's Rule is a powerful technique for evaluating limits that result in indeterminate forms $0/0$ or ∞/∞ . If $\lim_{x \rightarrow c} f(x) = 0$ and $\lim_{x \rightarrow c} g(x) = 0$, OR if $\lim_{x \rightarrow c} f(x) = \pm\infty$ and $\lim_{x \rightarrow c} g(x) = \pm\infty$, then: $\lim_{x \rightarrow c} [f(x)/g(x)] = \lim_{x \rightarrow c} [f'(x)/g'(x)]$ provided the limit on the right side exists (or is $\pm\infty$). You can apply the rule repeatedly if the indeterminate form persists.

Example 1: $\lim_{x \rightarrow 0} (\sin(x)/x)$ This is $0/0$. Apply L'Hopital's Rule: $\lim_{x \rightarrow 0} (\cos(x)/1) = \cos(0)/1 = 1/1 = 1$.

Example 2: $\lim_{x \rightarrow \infty} (e^x/x^2)$ This is ∞/∞ . Apply L'Hopital's Rule: $\lim_{x \rightarrow \infty} (e^x/(2x))$ Still ∞/∞ . Apply L'Hopital's Rule again: $\lim_{x \rightarrow \infty} (e^x/2) = \infty$.

Other indeterminate forms exist: $0 \cdot \infty$, $\infty - \infty$, 1^∞ , 0^∞ , ∞^0 . These can often be converted into $0/0$ or ∞/∞ forms using algebraic manipulation or logarithms. For $0 \cdot \infty$, rewrite $f(x)g(x)$ as $f(x)/(1/g(x))$ or $g(x)/(1/f(x))$. For forms involving exponents (1^∞ , 0^∞ , ∞^0), take the natural logarithm of the expression, evaluate the limit of the logarithm, then exponentiate the result.

Example 3: $\lim_{x \rightarrow 0^+} (x \ln(x))$ (form $0 \cdot (-\infty)$) Rewrite as $\lim_{x \rightarrow 0^+} (\ln(x)/(1/x))$. This is $(-\infty)/\infty$. Apply L'Hopital's Rule: $\lim_{x \rightarrow 0^+} [(1/x)/(-1/x^2)] = \lim_{x \rightarrow 0^+} (-x) = 0$.

Introduction to Antidifferentiation

Covers: Lec 27

This section introduces the inverse operation of differentiation, known as antidifferentiation or integration. Understanding this concept is fundamental for solving many problems in science and engineering, including calculating areas, volumes, and total change.

Antiderivatives and the Indefinite Integral

An **antiderivative** of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. For example, if $f(x) = 2x$, then $F(x) = x^2$ is an antiderivative because $(d/dx)(x^2) = 2x$. However, $x^2 + 5$ is also an antiderivative, as is $x^2 - 10$. In general, if $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$ (where C is any constant) is also an antiderivative. This is because the derivative of a constant is zero.

The collection of all antiderivatives of $f(x)$ is called the **indefinite integral** of $f(x)$, denoted by: $\int f(x) dx = F(x) + C$. Here, $\int$ is the integral sign, $f(x)$ is the integrand, dx indicates that x is the variable of integration, and C is the **constant of integration**.

Basic Antidifferentiation Rules

Just as there are rules for differentiation, there are corresponding rules for antidifferentiation.

- **Power Rule for Integration:** For any real number $n \neq -1$: $\int x^n dx = (x^{(n+1)})/(n+1) + C$
Example: $\int x^3 dx = (x^{(3+1)})/(3+1) + C = x^4/4 + C$
- **Special Case ($n = -1$):** $\int (1/x) dx = \ln|x| + C$
- **Constant Multiple Rule:** $\int c f(x) dx = c \int f(x) dx$

- **Sum/Difference Rule:** $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$
- **Antiderivatives of Trigonometric Functions:**
 - $\int \cos(x) dx = \sin(x) + C$
 - $\int \sin(x) dx = -\cos(x) + C$
 - $\int \sec^2(x) dx = \tan(x) + C$
- **Antiderivatives of Exponential Functions:**
 - $\int e^x dx = e^x + C$
 - $\int a^x dx = (a^x)/(\ln(a)) + C$

u-Substitution

U-substitution is a technique for finding antiderivatives that are the result of the Chain Rule in differentiation. It simplifies complex integrals by transforming them into simpler forms. The goal is to identify a part of the integrand, say u , whose derivative du is also present (or can be made present with a constant multiple).

Here are the steps for u-substitution:

1. **Choose u :** Select a suitable expression for u , usually an "inside" function or a part whose derivative simplifies the integral.
2. **Calculate du :** Find the differential $du = u'(x) dx$.
3. **Substitute:** Replace u and dx (or $u'(x) dx$) in the original integral with u and du . The integral should now be entirely in terms of u .
4. **Integrate:** Evaluate the new integral with respect to u .
5. **Substitute Back:** Replace u with its original expression in terms of x to get the final answer.

Example: $\int (2x+1)^3 dx$

1. Let $u = 2x + 1$.
2. Then $du = (d/dx)(2x + 1) dx = 2 dx$.
3. From $du = 2 dx$, we have $dx = (1/2) du$.
4. Substitute into the integral: $\int u^3 (1/2) du = (1/2) \int u^3 du$
5. Integrate with the Power Rule: $(1/2) * (u^4/4) + C = u^4/8 + C$
6. Substitute back $u = 2x + 1$: $(2x+1)^4/8 + C$

Common Mistake: Forgetting to include the constant of integration C . This is crucial because it represents the entire family of antiderivatives. Another common error is not correctly handling du , especially if a constant factor is missing and needs to be introduced (and compensated for).

KEY TAKEAWAYS

- Derivatives can determine rates of change between related quantities, approximate function values, and iteratively find roots of equations.
- The Mean Value Theorem connects the average rate of change over an interval to an instantaneous rate of change at some point within that interval.
- First and second derivatives are essential tools for analyzing a function's increasing/decreasing behavior, concavity, and local extrema, enabling accurate curve sketching.
- Optimization problems use derivatives to find the absolute maximum or minimum values of a function, solving real-world challenges under given constraints.
- L'Hopital's Rule provides a method for evaluating limits of indeterminate forms like $0/0$ or ∞/∞ by taking derivatives of the numerator and denominator.
- Antidifferentiation is the inverse process of differentiation, yielding a family of functions (the indefinite integral) that differ by a constant.
- U-substitution simplifies complex integrals by reversing the Chain Rule, making them easier to integrate using basic rules.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. Explain the purpose of Newton's Method and describe a scenario where it might fail to converge.
2. How do the first and second derivatives help in sketching the graph of a function?
3. What is an indeterminate form, and when should you use L'Hopital's Rule?
4. What is the significance of the constant of integration C in an indefinite integral?
5. Describe the main idea behind u-substitution in integration.

CHAPTER 4

Techniques of Integration and Infinite Series

This chapter extends your understanding of calculus beyond basic differentiation and integration. You will learn powerful techniques to integrate a wider range of functions, including those involving products, trigonometric expressions, and rational functions. Mastery of these methods is crucial for solving complex problems in engineering, physics, and other sciences.

Beyond integration, this chapter introduces the fundamental concepts of sequences and infinite series. You will explore how to determine if an infinite sum converges to a finite value and learn various tests to assess this convergence. These topics form the bedrock for advanced mathematical concepts, including power series and Taylor series, which are essential for approximating functions and solving differential equations.

Integration by Parts and Trigonometric Integrals

Covers: Lec 28 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 29 | MIT 18.01 Single Variable Calculus, Fall 2007

Many integrals cannot be solved using simple substitution or direct formulas. **Integration by Parts** is a technique used to integrate products of functions. It derives from the product rule for differentiation. The formula is:

$$\int u \, dv = uv - \int v \, du$$

The key is to choose u and dv carefully. A common heuristic for choosing u is the **LIATE** rule, which prioritizes functions in this order:

- Logarithmic functions (e.g., $\ln(x)$)
- Inverse trigonometric functions (e.g., $\arctan(x)$)
- Algebraic functions (e.g., x^n)
- Trigonometric functions (e.g., $\sin(x)$)
- Exponential functions (e.g., e^x)

Functions higher on the list are generally chosen as u , with the remaining part as dv . This choice often simplifies the integral $\int v \, du$.

Example: Integration by Parts

To evaluate $\int x e^x \, dx$:

1. Choose $u = x$ (algebraic) and $dv = e^x \, dx$ (exponential).
2. Compute $du = dx$ and $v = \int e^x \, dx = e^x$.
3. Apply the formula: $\int x e^x \, dx = x e^x - \int e^x \, dx$

4. Evaluate the remaining integral: $\int x e^x dx = x e^x - e^x + C$

Trigonometric Integrals involve powers of trigonometric functions. Specific strategies apply depending on the powers involved.

Integrals of $\sin^m(x)\cos^n(x)$

- If n is odd, save one $\cos(x)$ and convert the remaining $\cos^2(x)$ to $1 - \sin^2(x)$. Let $u = \sin(x)$.
- If m is odd, save one $\sin(x)$ and convert the remaining $\sin^2(x)$ to $1 - \cos^2(x)$. Let $u = \cos(x)$.
- If both m and n are even, use half-angle identities: $\sin^2(x) = (1 - \cos(2x))/2$ and $\cos^2(x) = (1 + \cos(2x))/2$.

Example: $\int \sin^3(x) dx$

1. $\int \sin^2(x) \sin(x) dx$
2. $\int (1 - \cos^2(x)) \sin(x) dx$
3. Let $u = \cos(x)$, so $du = -\sin(x) dx$.
4. $\int (1 - u^2) (-du) = \int (u^2 - 1) du$
5. $= u^3/3 - u + C = \cos^3(x)/3 - \cos(x) + C$

Integrals of $\tan^m(x)\sec^n(x)$

- If n is even (and $n \geq 2$), save $\sec^2(x)$ and convert remaining $\sec^2(x)$ to $1 + \tan^2(x)$. Let $u = \tan(x)$.
- If m is odd (and $m \geq 1$), save $\sec(x)\tan(x)$ and convert remaining $\tan^2(x)$ to $\sec^2(x) - 1$. Let $u = \sec(x)$.

Practice is essential for mastering these techniques. Try $\int x^2 \sin(x) dx$ for integration by parts, which requires applying the formula twice. For trigonometric integrals, try $\int \cos^4(x) dx$ and $\int \tan^2(x) \sec^4(x) dx$.

Trigonometric Substitution, Partial Fractions, and Improper Integrals

Covers: Lec 30 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 31 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 32 | MIT 18.01 Single Variable Calculus, Fall 2007

Trigonometric Substitution is a technique for integrals involving expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$. It converts the integral into a trigonometric integral, which can then be solved using the methods from the previous section.

Cases for Trigonometric Substitution

1. **$\sqrt{a^2 - x^2}$** : Let $x = a \sin(\theta)$. Then $dx = a \cos(\theta) d\theta$ and $\sqrt{a^2 - x^2} = a \cos(\theta)$.
2. **$\sqrt{a^2 + x^2}$** : Let $x = a \tan(\theta)$. Then $dx = a \sec^2(\theta) d\theta$ and $\sqrt{a^2 + x^2} = a \sec(\theta)$.
3. **$\sqrt{x^2 - a^2}$** : Let $x = a \sec(\theta)$. Then $dx = a \sec(\theta)\tan(\theta) d\theta$ and $\sqrt{x^2 - a^2} = a \tan(\theta)$.

After integrating, convert back to the original variable using a reference triangle.

Example: $\int 1 / (x^2 \sqrt{9 - x^2}) dx$

1. This matches $\sqrt{a^2 - x^2}$ with $a = 3$. Let $x = 3 \sin(\theta)$.
2. Then $dx = 3 \cos(\theta) d\theta$ and $\sqrt{9 - x^2} = \sqrt{9 - 9 \sin^2(\theta)} = 3 \cos(\theta)$.
3. Substitute: $\int (3 \cos(\theta) d\theta) / ((3 \sin(\theta))^2 (3 \cos(\theta))) = \int 1 / (9 \sin^2(\theta)) d\theta = (1/9) \int \csc^2(\theta) d\theta = (-1/9) \cot(\theta) + C$
4. From $x = 3 \sin(\theta)$, $\sin(\theta) = x/3$. Draw a right triangle with opposite x and hypotenuse 3. The adjacent side is $\sqrt{9 - x^2}$.
5. $\cot(\theta) = \text{adjacent} / \text{opposite} = \sqrt{9 - x^2} / x$.
6. Result: $(-1/9) (\sqrt{9 - x^2} / x) + C$.

Partial Fraction Decomposition is used to integrate rational functions (polynomials divided by polynomials). The method involves breaking down a complex rational function into a sum of simpler fractions that are easier to integrate. This works when the degree of the numerator is less than the degree of the denominator. If not, perform polynomial long division first.

Steps for Partial Fractions

1. Factor the denominator completely into linear and irreducible quadratic factors.
2. For each distinct linear factor $(ax + b)$, include a term $A / (ax + b)$.
3. For each repeated linear factor $(ax + b)^k$, include terms $A_1 / (ax + b) + A_2 / (ax + b)^2 + \dots + A_k / (ax + b)^k$.
4. For each irreducible quadratic factor $(ax^2 + bx + c)$, include a term $(Ax + B) / (ax^2 + bx + c)$.
5. Set the original fraction equal to the sum of these partial fractions. Multiply by the common denominator to clear it.
6. Solve for the unknown constants (A, B, etc.) by equating coefficients or choosing convenient values for x .

Example: $\int 1 / ((x - 1)(x + 2)) dx$

1. $1 / ((x - 1)(x + 2)) = A / (x - 1) + B / (x + 2)$
2. $1 = A(x + 2) + B(x - 1)$
3. If $x = 1$, $1 = A(3) + B(0) \Rightarrow A = 1/3$.
4. If $x = -2$, $1 = A(0) + B(-3) \Rightarrow B = -1/3$.
5. $\int (1/3) / (x - 1) dx + \int (-1/3) / (x + 2) dx$
6. $= (1/3) \ln|x - 1| - (1/3) \ln|x + 2| + C = (1/3) \ln|(x - 1) / (x + 2)| + C$

Improper Integrals are definite integrals where either one or both limits of integration are infinite, or the integrand has an infinite discontinuity within the interval of integration. They are evaluated using limits.

Types of Improper Integrals

1. Infinite Limits:

- $\int_a^\infty f(x) \, dx = \lim_{(b \rightarrow \infty)} \int_a^b f(x) \, dx$
- $\int_{-\infty}^b f(x) \, dx = \lim_{(a \rightarrow -\infty)} \int_a^b f(x) \, dx$
- $\int_{-\infty}^\infty f(x) \, dx = \int_{-\infty}^c f(x) \, dx + \int_c^\infty f(x) \, dx$ (where c is any real number)

2. Infinite Discontinuities: If $f(x)$ has a discontinuity at b (an endpoint):

- $\int_a^b f(x) \, dx = \lim_{(t \rightarrow b^-)} \int_a^t f(x) \, dx$
- If $f(x)$ has a discontinuity at c within $[a, b]$: $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$

An improper integral **converges** if the limit exists and is a finite number. Otherwise, it **diverges**.

Example: $\int_1^\infty 1/x^2 \, dx$

1. $\lim_{(b \rightarrow \infty)} \int_1^b x^{-2} \, dx$
2. $\lim_{(b \rightarrow \infty)} [-x^{-1}]_1^b$
3. $\lim_{(b \rightarrow \infty)} (-1/b - (-1/1))$
4. $\lim_{(b \rightarrow \infty)} (-1/b + 1) = 0 + 1 = 1.$

The integral converges to 1. Try $\int_0^1 1/\sqrt{x} \, dx$ for an improper integral with a discontinuity.

Introduction to Sequences and Series

Covers: Lec 33 | MIT 18.01 Single Variable Calculus, Fall 2007

This section introduces sequences and series, which are fundamental concepts for understanding infinite sums and their applications.

A **sequence** is an ordered list of numbers, often defined by a formula for its n -th term, denoted as $\{a_n\}$. For example, the sequence $\{1/n\}$ is 1, $1/2$, $1/3$, $1/4 \dots$

Convergence of Sequences

A sequence $\{a_n\}$ **converges** to a limit L if, as n approaches infinity, the terms of the sequence get arbitrarily close to L . We write $\lim_{(n \rightarrow \infty)} a_n = L$. If the limit does not exist or is infinite, the sequence **diverges**.

Example: Convergence of a Sequence

Consider the sequence $\{(2n + 1) / (n + 5)\}$. To find its limit:

$$\lim_{(n \rightarrow \infty)} (2n + 1) / (n + 5) = \lim_{(n \rightarrow \infty)} (2 + 1/n) / (1 + 5/n) = (2 + 0) / (1 + 0) = 2$$

The sequence converges to 2.

A **series** is the sum of the terms of a sequence. An **infinite series** is the sum of an infinite number of terms, written as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$.

Convergence of Series

Determining if an infinite series converges to a finite sum is a central problem. We define the **partial sum** S_N as the sum of the first N terms: $S_N = \sum_{n=1}^N a_n$.

An infinite series **converges** if the sequence of its partial sums $\{S_N\}$ converges to a finite limit S . That is, $\lim_{N \rightarrow \infty} S_N = S$. If the limit does not exist or is infinite, the series **diverges**.

The Nth Term Test for Divergence

If $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum a_n$ diverges. If $\lim_{n \rightarrow \infty} a_n = 0$, the test is inconclusive; the series might converge or diverge.

Geometric Series

A **geometric series** has the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$. It converges if $|r| < 1$, and its sum is $a / (1 - r)$. If $|r| \geq 1$, the series diverges.

Example: Geometric Series

The series $\sum_{n=0}^{\infty} (1/2)^n = 1 + 1/2 + 1/4 + \dots$ is a geometric series with $a = 1$ and $r = 1/2$. Since $|1/2| < 1$, it converges to $1 / (1 - 1/2) = 1 / (1/2) = 2$.

Harmonic Series

The **harmonic series** $\sum_{n=1}^{\infty} 1/n = 1 + 1/2 + 1/3 + \dots$ is a classic example of a divergent series, even though its terms approach zero. This highlights why the Nth Term Test is inconclusive when the limit is zero.

Practice determining the convergence of sequences like $\{n^2 / (n^2 + 1)\}$ and $\{\cos(n\pi)\}$. For series, try to identify if they are geometric and find their sum if they converge, or use the Nth Term Test for divergence.

Convergence Tests for Series

Covers: Lec 35 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 36 | MIT 18.01 Single Variable Calculus, Fall 2007

Since directly finding the sum of partial sums is often impossible, various tests help determine if an infinite series converges or diverges. This section covers several important convergence tests.

The Integral Test

If f is a positive, continuous, and decreasing function for $x \geq 1$, and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x) dx$ either both converge or both diverge.

Example: Integral Test

Consider the p -series $\sum_{n=1}^{\infty} 1/n^p$. If $p > 1$, the series converges. If $p \leq 1$, it diverges. For $p = 2$, $\sum_{n=1}^{\infty} 1/n^2$ converges because $\int_1^{\infty} 1/x^2 dx = 1$ (a finite value).

Comparison Tests

These tests compare a given series $\sum a_n$ (with positive terms) to a known series $\sum b_n$.

• Direct Comparison Test:

- If $0 \leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all n and $\sum b_n$ diverges, then $\sum a_n$ diverges.

• Limit Comparison Test:

- If $\lim_{n \rightarrow \infty} (a_n / b_n) = L$ where L is a finite, positive number ($L > 0$), then both $\sum a_n$ and $\sum b_n$ either converge or both diverge.
- This test is very useful when the direct comparison is difficult to apply.

Example: Limit Comparison Test

Consider $\sum_{n=1}^{\infty} (n+1) / (n^2 + 3)$. We can compare it to the harmonic series $\sum 1/n$, which diverges. Let $a_n = (n+1) / (n^2 + 3)$ and $b_n = 1/n$.

$$\begin{aligned} \lim_{n \rightarrow \infty} ((n+1) / (n^2 + 3)) / (1/n) &= \lim_{n \rightarrow \infty} (n^2 + n) / (n^2 + 3) \\ &= \lim_{n \rightarrow \infty} (1 + 1/n) / (1 + 3/n^2) = 1/1 = 1 \end{aligned}$$

Since the limit is 1 (a finite, positive number) and $\sum 1/n$ diverges, the series $\sum (n+1) / (n^2 + 3)$ also diverges.

Alternating Series Test

An **alternating series** has terms that alternate in sign, like $\sum (-1)^{(n-1)} b_n$ where $b_n > 0$. The Alternating Series Test states that if:

1. b_n is positive and decreasing (i.e., $b_{(n+1)} \leq b_n$ for all n), and
2. $\lim_{n \rightarrow \infty} b_n = 0$,

then the alternating series $\sum (-1)^{(n-1)} b_n$ converges.

Example: Alternating Series Test

The alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{(n-1)} (1/n)$ converges because $b_n = 1/n$ is positive, decreasing, and $\lim_{n \rightarrow \infty} 1/n = 0$.

Absolute and Conditional Convergence

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges. If a series is absolutely convergent, then it is also convergent.

A series $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

The alternating harmonic series is conditionally convergent because $\sum (-1)^{(n-1)} (1/n)$ converges, but $\sum |(-1)^{(n-1)} (1/n)| = \sum 1/n$ (the harmonic series) diverges.

Practice applying these tests to various series. For example, determine the convergence of $\sum_{n=1}^{\infty} (\sin(n) / n^3)$ (use comparison with $1/n^3$) and $\sum_{n=1}^{\infty} (-1)^n (n / (n^2 + 1))$ (use alternating series test).

KEY TAKEAWAYS

- Integration by parts is a technique for integrating products of functions, following the formula $\int u \, dv = uv - \int v \, du$.
- Trigonometric integrals require specific strategies based on the powers of sine, cosine, tangent, and secant functions.
- Trigonometric substitution is used for integrals involving $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 - a^2}$ by converting them into trigonometric forms.
- Partial fraction decomposition breaks down complex rational functions into simpler, integrable fractions.
- Improper integrals, with infinite limits or discontinuities, are evaluated using limits to determine convergence or divergence.
- Sequences are ordered lists of numbers, while series are the sums of the terms of a sequence.
- Various tests, including the Integral Test, Comparison Tests, and Alternating Series Test, help determine the convergence or divergence of infinite series.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. How do you choose u and dv when using integration by parts, and why is this choice important?
2. What are the three main cases for trigonometric substitution, and what substitution corresponds to each?
3. Explain the difference between a sequence and a series. How is the convergence of an infinite series defined?
4. When is the Nth Term Test for Divergence useful, and what are its limitations?
5. Describe the conditions for the Alternating Series Test to apply, and what it tells you about the series.

CHAPTER 5

Taylor and Maclaurin Series

This chapter introduces Taylor and Maclaurin series, powerful tools for representing functions as infinite polynomials. Understanding these series allows us to approximate complex functions, evaluate difficult limits, and solve integrals that are otherwise intractable. We will explore their construction, properties, and practical applications.

Mastering these concepts provides a deeper insight into function behavior and is fundamental for advanced mathematical and scientific computations. You will learn how to build these series, determine their convergence, and apply them to real problems, extending your calculus toolkit significantly.

Constructing Taylor and Maclaurin Series

Covers: Lec 37 | MIT 18.01 Single Variable Calculus, Fall 2007

Taylor series provide a way to represent a function as an infinite sum of terms. Each term is calculated from the function's derivatives at a single point. This point is called the center of the series. For a function $f(x)$ that has derivatives of all orders at a point a , its Taylor series is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\ = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$

Here, $f^{(n)}(a)$ denotes the n -th derivative of $f(x)$ evaluated at $x=a$. The term $n!$ is n factorial. When the series is centered at $a=0$, it is called a Maclaurin series. Maclaurin series are a special case of Taylor series and are very common due to their simpler form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

Example: Finding the Maclaurin Series for e^x

To find the Maclaurin series for $f(x) = e^x$, we need to find its derivatives and evaluate them at $x=0$.

1. $f(x) = e^x \implies f(0) = e^0 = 1$
2. $f(x) = e^x \implies f'(0) = e^0 = 1$
3. $f(x) = e^x \implies f''(0) = e^0 = 1$
4. $f(x) = e^x \implies f'''(0) = e^0 = 1$

In general, $f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$ for all $n \geq 0$.

Now, substitute these into the Maclaurin series formula:

$$\begin{aligned}
 e^x &= \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \\
 &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
 &= \sum_{n=0}^{\infty} \frac{x^n}{n!}
 \end{aligned}$$

This series converges for all real numbers x . This means that for any x , the infinite sum equals e^x . Truncating the series after a few terms gives a polynomial approximation of e^x .

Common Maclaurin Series

It is helpful to memorize or be able to quickly derive some common Maclaurin series:

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ (for all x)
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ (for all x)
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ (for all x)
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ (for $|x| < 1$)

Common Mistakes

One frequent error is forgetting the $n!$ in the denominator of the Taylor series formula. Another is incorrectly evaluating the derivatives or the function itself at the center point a . Always double-check these initial calculations.

Convergence and Advanced Applications of Taylor Series

Covers: Lec 38 | MIT 18.01 Single Variable Calculus, Fall 2007 · Lec 39 | MIT 18.01 Single Variable Calculus, Fall 2007

A Taylor series does not always converge for all values of x . The set of x values for which the series converges is called the interval of convergence. The radius of convergence, R , describes how far from the center a the series converges. For a series centered at a , the interval of convergence is typically $(a-R, a+R)$, possibly including one or both endpoints. The Ratio Test is commonly used to find R .

Example: Radius of Convergence for e^x

For the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we apply the Ratio Test. Let $a_n = \frac{x^n}{n!}$.

$$\begin{aligned}
 L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = |x| \lim_{n \rightarrow \infty} \frac{1}{n+1} = |x| \cdot 0 = 0
 \end{aligned}$$

Since $L=0$ is always less than 1, the series converges for all x . Thus, the radius of convergence is $R=\infty$.

Taylor Polynomials and Approximations

When we truncate a Taylor series after a finite number of terms, we get a Taylor polynomial, $P_N(x)$, which approximates the function $f(x)$. The higher the degree N of the polynomial, the better the approximation, especially near the center a . For instance, the N -th degree Taylor polynomial for $f(x)$ centered at a is:

$$P_N(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

To estimate the error in this approximation, we use the Lagrange Remainder (also known as Taylor's Theorem with Remainder). The remainder $R_N(x)$ is the difference between $f(x)$ and $P_N(x)$, and it is given by:

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c between a and x . To find an upper bound for the error, we find the maximum possible value of $|f^{(N+1)}(c)|$ on the interval between a and x .

Example: Approximating $e^{0.1}$ and Estimating Error

Use a 3rd degree Maclaurin polynomial to approximate $e^{0.1}$ and estimate the error. For $f(x) = e^x$, $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

$$\begin{aligned} P_3(0.1) &= 1 + 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} \\ &= 1 + 0.1 + 0.005 + 0.0001666\dots \\ &\approx 1.1051667 \end{aligned}$$

To estimate the error, we use $R_3(0.1) = \frac{f^{(4)}(c)}{4!}(0.1)^4$. Since $f^{(4)}(x) = e^x$, $R_3(0.1) = \frac{e^c}{24}(0.1)^4$. For c between 0 and 0.1 , $e^c < e^{0.1}$. We know $e^{0.1} < e^1 \approx 2.718$. A safer upper bound for e^c for $c \in [0, 0.1]$ is $e^{0.1} < e^{0.5} = \sqrt{e} \approx 1.65$. Let's use $e^{0.1} < 1.11$ (a very close approximation, or simply use $e^1 = 2.718$ for a looser but safe bound).

Using $e^c < e^{0.1} \approx 1.10517$, the error bound is:

$$|R_3(0.1)| < \frac{1.10517}{24}(0.0001) \approx 0.0000046$$

This shows the approximation is very accurate.

Applications of Taylor Series

Taylor series are invaluable for:

- **Evaluating Limits:** Complex limits can often be simplified by replacing functions with their Taylor series expansions. For example, to evaluate $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$, substitute the series for $\sin x$:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots) - x}{x^3} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^3} \\ &= \lim_{x \rightarrow 0} \left(-\frac{1}{3!} + \frac{x^2}{5!} - \dots \right) = -\frac{1}{6} \end{aligned}$$

- **Approximating Integrals:** Many functions do not have elementary antiderivatives. Their definite integrals can be approximated by integrating their Taylor series term by term. For example, $\int_0^1 \sin(x^2) dx$ can be approximated using the series for $\sin(x^2)$.
- **Binomial Series:** A special case of Maclaurin series for $(1+x)^k$ is the binomial series:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \dots$$

This series converges for $|x| < 1$. It is very useful for approximating roots and other fractional powers.

Practice constructing Taylor series for various functions, determining their radii of convergence, and using Taylor polynomials to approximate values and estimate errors. Also, practice using series to evaluate limits and approximate integrals.

KEY TAKEAWAYS

- Taylor series represent functions as infinite polynomials derived from the function's derivatives at a specific center point.
- Maclaurin series are a special type of Taylor series centered at zero, simplifying their form.
- The radius and interval of convergence define the range of x -values for which a Taylor series accurately represents the original function.
- Taylor polynomials are finite truncations of Taylor series, providing effective approximations of functions near the series' center.
- The Lagrange Remainder provides a method to estimate the maximum error when using a Taylor polynomial for approximation.
- Taylor series are powerful tools for evaluating complex limits and approximating definite integrals of functions without elementary antiderivatives.
- The binomial series is a specific Maclaurin series for functions of the form $(1+x)^k$, useful for fractional powers and roots.

CHECK YOUR UNDERSTANDING

Answer these before moving on. Model answers are in the Answer Key at the back of this guide.

1. Define Taylor series and Maclaurin series, highlighting their key difference.
2. How do you determine the coefficients of a Taylor series?
3. Explain the significance of the radius of convergence for a Taylor series.

4. How can Taylor polynomials be used to approximate function values, and what is the role of the degree of the polynomial?
5. What is the purpose of the Lagrange Remainder in the context of Taylor series approximations?

REFERENCE

Glossary

Antiderivative

A function $F(x)$ whose derivative is $f(x)$, meaning $F'(x) = f(x)$.

Binomial series

The Maclaurin series for the function $(1+x)^k$, which is valid for $|x| < 1$.

Chain rule

A rule for differentiating composite functions, stating $\frac{d}{dx} [f(g(x))] = f'(g(x)) * g'(x)$.

Concavity

A description of the curvature of a function's graph; concave up means the graph opens upward, and concave down means it opens downward, determined by the sign of the second derivative.

Convergence

A property of a sequence or series where its terms approach a finite limit or its sum approaches a finite value, respectively.

Critical point

A point in the domain of a function where its first derivative is either zero or undefined; these points are candidates for local maxima or minima.

Derivative

The instantaneous rate of change of a function with respect to its independent variable, representing the slope of the tangent line.

Difference quotient

The expression $[f(x+h) - f(x)] / h$, whose limit as h approaches zero defines the derivative.

Implicit differentiation

A technique for differentiating equations that implicitly define a function, without explicitly solving for one variable in terms of the other.

Improper integral

A definite integral where at least one limit of integration is infinite, or the integrand has an infinite discontinuity within the interval.

Inflection point

A point on the graph of a function where the concavity changes (from concave up to concave down, or vice versa).

Integration by parts

A technique for integrating products of functions using the formula $\int u \, dv = uv - \int v \, du$.

Interval of convergence

The set of all x -values for which a power series converges, typically given as an interval $(a-R, a+R)$ possibly including endpoints.

L'hopital's rule

A rule used to evaluate limits of indeterminate forms $0/0$ or ∞/∞ by taking the ratio of the derivatives of the numerator and denominator.

Lagrange remainder

A formula that provides an upper bound for the error when approximating a function with a Taylor polynomial.

Limit

The value that a function or sequence approaches as the input or index approaches some value.

Linear approximation

Using the tangent line to a function at a point a to approximate the function's value at a nearby point x .

Linear approximation (linearization)

Using the tangent line to a function's graph at a specific point to estimate the function's values near that point.

Maclaurin series

A special case of a Taylor series where the series is centered at zero.

Mean value theorem

A theorem stating that for a continuous and differentiable function on an interval, there is a point where the instantaneous rate of change equals the average rate of change.

Newton's method

An iterative numerical method for finding successively better approximations to the roots (or zeros) of a real-valued function.

Optimization

The process of finding the maximum or minimum value of a function, often in a real-world context, by using derivative tests on critical points within a specified domain.

Partial fraction decomposition

A method to rewrite a rational function as a sum of simpler fractions, making it easier to integrate.

Radius of convergence

A value R that defines the range around the series' center for which the series converges; if R is infinite, it converges everywhere.

Related rates

Problems that involve finding the rate of change of one quantity by relating it to other quantities whose rates of change are known.

Sequence

An ordered list of numbers, typically defined by a formula for its n -th term.

Series

The sum of the terms of a sequence, which can be finite or infinite.

Taylor polynomial

A finite sum of the first N terms of a Taylor series, used to approximate the function.

Taylor series

An infinite series representation of a function as a sum of terms calculated from the function's derivatives at a single point.

Trigonometric substitution

A method for simplifying integrals containing expressions like $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ by substituting trigonometric functions.

REFERENCE

Answer Key

Chapter 1: Introduction to Differentiation

1. **Explain the relationship between the slope of a secant line and the slope of a tangent line.**

The slope of a secant line represents the average rate of change between two points on a curve. As the two points on the secant line get infinitesimally close to each other, the secant line approaches the tangent line. The slope of the tangent line then represents the instantaneous rate of change at that single point, which is the derivative.

2. **State the product rule and quotient rule for differentiation.**

The product rule states that $\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. The quotient rule states that $\frac{d}{dx} [f(x)/g(x)] = [f'(x)g(x) - f(x)g'(x)] / [g(x)]^2$. These rules are used to differentiate functions that are products or ratios of other functions.

3. **When is the chain rule necessary, and how is it applied?**

The chain rule is necessary when differentiating composite functions, which are functions within functions, like $f(g(x))$. It is applied by differentiating the "outside" function with respect to its argument (the "inside" function), and then multiplying the result by the derivative of the "inside" function.

4. **Describe a scenario where implicit differentiation would be required.**

Implicit differentiation is required when an equation defines a relationship between x and y but does not explicitly express y as a function of x . For example, finding dy/dx for the equation of a circle, $x^2 + y^2 = r^2$, necessitates implicit differentiation because y cannot be uniquely written as $f(x)$.

5. **What are the derivatives of $\sin(x)$ and $\cos(x)$?**

The derivative of $\sin(x)$ with respect to x is $\cos(x)$. The derivative of $\cos(x)$ with respect to x is $-\sin(x)$. These are fundamental trigonometric derivatives used extensively in calculus.

Chapter 2: Applications of Differentiation

1. **What is the key step in solving a related rates problem, and why is it important not to substitute numerical values too early?**

The key step is implicitly differentiating the equation that relates the variables with respect to time. It is crucial not to substitute numerical values for variables that are changing before differentiation. If you substitute too early, those variables become constants, and their derivatives would incorrectly be zero, leading to an incorrect result.

2. **Describe the primary role of the first derivative and the second derivative in curve sketching.**

The first derivative, $f'(x)$, indicates where a function is increasing or decreasing and helps identify local

maximum and minimum points. The second derivative, $f''(x)$, reveals the concavity of the function (concave up or down) and helps locate inflection points, where the concavity changes.

3. When solving an optimization problem on a closed interval, what points must you evaluate to find the absolute maximum and minimum?

To find the absolute maximum and minimum of a function on a closed interval, you must evaluate the function at all critical points that lie within the interval. You must also evaluate the function at the endpoints of the interval. The largest and smallest of these values will be the absolute maximum and minimum, respectively.

4. Explain the concept behind linear approximation and provide its formula.

Linear approximation uses the tangent line to a function's graph at a known point to estimate the function's values near that point. The tangent line provides the best linear fit to the curve locally. The formula for the linear approximation $L(x)$ of $f(x)$ at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$.

5. What is Newton's Method used for, and what information does it require to proceed iteratively?

Newton's Method is an iterative numerical technique used to find successively better approximations to the roots (or zeros) of a real-valued function. To proceed iteratively, it requires an initial guess for the root, the function itself $f(x)$, and its first derivative $f'(x)$.

Chapter 3: Applications of Derivatives and Introduction to Antidifferentiation

1. Explain the purpose of Newton's Method and describe a scenario where it might fail to converge.

Newton's Method is used to find numerical approximations for the roots of a function. It works by repeatedly finding the x-intercept of the tangent line to the function. It might fail to converge if the initial guess is too far from a root, if the derivative $f'(x)$ is zero or very small near a root, or if the function oscillates wildly.

2. How do the first and second derivatives help in sketching the graph of a function?

The first derivative $f'(x)$ indicates where the function is increasing or decreasing and helps identify local maxima and minima. The second derivative $f''(x)$ reveals the concavity of the function (concave up or down) and helps locate inflection points. Together, these derivatives provide a comprehensive picture of the function's shape.

3. What is an indeterminate form, and when should you use L'Hopital's Rule?

An indeterminate form is a limit expression that does not immediately reveal its value, such as $0/0$ or ∞/∞ . You should use L'Hopital's Rule when evaluating a limit that results in one of these indeterminate forms after direct substitution. The rule allows you to take the derivatives of the numerator and denominator separately.

4. What is the significance of the constant of integration C in an indefinite integral?

The constant of integration C represents the entire family of antiderivatives for a given function. Since the derivative of any constant is zero, $F(x) + C$ will always have the same derivative $f(x)$ as $F(x)$. It signifies that there are infinitely many antiderivatives, differing only by a vertical shift.

5. Describe the main idea behind u-substitution in integration.

U-substitution is a technique to simplify integrals that resemble the result of the Chain Rule. It involves identifying an "inner" function u and its differential du within the integrand. By substituting u and du , the integral is transformed into a simpler form that can be integrated using basic rules, then u is substituted back.

Chapter 4: Techniques of Integration and Infinite Series

1. How do you choose u and dv when using integration by parts, and why is this choice important?

The LIATE rule (Logarithmic, Inverse trig, Algebraic, Trigonometric, Exponential) is a useful heuristic for choosing u . Functions higher on this list are generally chosen as u . This choice is important because it aims to simplify the integral $\int v \, du$, making it easier to evaluate than the original integral.

2. What are the three main cases for trigonometric substitution, and what substitution corresponds to each?

The three cases are $\sqrt{a^2 - x^2}$ (use $x = a \sin(\theta)$), $\sqrt{a^2 + x^2}$ (use $x = a \tan(\theta)$), and $\sqrt{x^2 - a^2}$ (use $x = a \sec(\theta)$). Each substitution is chosen to simplify the radical expression using a Pythagorean identity.

3. Explain the difference between a sequence and a series. How is the convergence of an infinite series defined?

A sequence is an ordered list of numbers, such as $\{1, 1/2, 1/3, \dots\}$. A series is the sum of the terms of a sequence, such as $1 + 1/2 + 1/3 + \dots$. An infinite series converges if the limit of its sequence of partial sums exists and is a finite number.

4. When is the Nth Term Test for Divergence useful, and what are its limitations?

The Nth Term Test for Divergence is useful when $\lim_{n \rightarrow \infty} a_n \neq 0$, in which case the series $\sum a_n$ diverges. Its limitation is that if $\lim_{n \rightarrow \infty} a_n = 0$, the test is inconclusive; the series might still converge or diverge, requiring other tests.

5. Describe the conditions for the Alternating Series Test to apply, and what it tells you about the series.

The Alternating Series Test applies to series of the form $\sum (-1)^{n-1} b_n$ (or $\sum (-1)^n b_n$) where $b_n > 0$. The conditions are that b_n must be decreasing and $\lim_{n \rightarrow \infty} b_n = 0$. If these conditions are met, the test tells you that the alternating series converges.

Chapter 5: Taylor and Maclaurin Series

1. Define Taylor series and Maclaurin series, highlighting their key difference.

A Taylor series represents a function as an infinite polynomial based on its derivatives at a specific point, called the center. A Maclaurin series is a special case of a Taylor series where the center point is specifically zero. Both use the same general formula, but the Maclaurin series simplifies when the center is zero.

2. How do you determine the coefficients of a Taylor series?

The coefficients of a Taylor series are determined by the formula $f^{(n)}(a) / n!$. This means you need to find the n -th derivative of the function, evaluate it at the series' center point ' a ', and then divide by n factorial. This process is repeated for each term in the series.

3. Explain the significance of the radius of convergence for a Taylor series.

The radius of convergence, R , indicates how far from the series' center the Taylor series accurately represents the original function. Within the interval $(a-R, a+R)$, the series converges to the function's value. Outside this interval, the series diverges and does not represent the function.

4. How can Taylor polynomials be used to approximate function values, and what is the role of the degree of the polynomial?

Taylor polynomials are finite sums of Taylor series terms, providing polynomial approximations of functions. The higher the degree of the polynomial, the more terms are included, generally leading to a more accurate approximation of the function, especially closer to the series' center. These approximations are useful for calculations or when the exact function is hard to work with.

5. What is the purpose of the Lagrange Remainder in the context of Taylor series approximations?

The Lagrange Remainder provides a way to quantify the error when using a Taylor polynomial to approximate a function. It gives an upper bound for the absolute difference between the true function value and the polynomial approximation. This allows us to know the maximum possible error and assess the accuracy of our approximation.

CLOSING

Where to Go Next

Upon completing "MIT 18.01: Single Variable Calculus," you will have built a powerful analytical toolkit. To consolidate your skills, revisit challenging problems and work through additional exercises from textbooks or online resources. Consider applying calculus to problems in physics or engineering to see its practical utility.

You can also explore related mathematical disciplines. Many students progress to **Multivariable Calculus**, which extends these concepts to higher dimensions. **Differential Equations** is another natural progression, focusing on equations involving derivatives.

Further studies might include **Linear Algebra**, essential for many scientific and computational fields. Your mastery of single variable calculus provides an excellent foundation for these advanced subjects and prepares you for continued academic or professional growth.



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